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LABORATORY OF PLASMA STUDIES
CORNELL UNIVERSITY
ITHACA, NEW YORK

PROPAGATION OF HIGH CURRENT RELATIVISTIC ELECTRON BEAMS

by

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LPS 16 JUNE 1969

Supported by the Office of Naval Research under Contract No. N00014-67-A-0077-0002.

Major portions of this report are reproduced from a Ph.D Thesis by D. A. Hammer, Cornell University, June, 1969.

ABSTRACT

Theoretical self-consistent beam models have been developed which allow the propagation of relativistic electron fluxes in excess of the Alfvén-Lawson critical-current limit for a fully neutralized beam. Development of a simple, fully relativistic, self-consistent equilibrium is described which can carry arbitrarily large currents at or near complete electrostatic neutralization. A discussion of a model for magnetic neutralization is presented wherein it is shown that large numbers of electrons from a background plasma are counter-streaming slowly within the beam so that the net current density in the system, and therefore the magnetic field, is nearly zero. A solution of an initial-value problem for a beam-plasma system is given which indicates that magnetic neutralization can be expected to occur for plasma densities that are large compared to beam densities. It is found that the application of a strong axial magnetic field to a uniform beam allows propagation regardless of the magnitude of the beam current.

I. INTRODUCTION

Theoretical interest in relativistic electron beams began with W. H. Bennett's 1934 paper[1] in which he pointed out that electrostatically neutralized high current, electron streams can be magnetically self-focusing. H. Alfvén[2] was motivated to consider charged particle beams in order to explain certain observations concerning cosmic rays. He derived an upper limit to the possible current of cosmic rays that can propagate through space in a given direction. His model was a cylindrically symmetric, monoenergetic, uniform current density stream of identical particles, and he assumed that the ionized matter in interstellar space would insure electrical neutralization. The current limit, I_A, which Alfvén derived is due to the pinch forces of the beam's self-magnetic field, and is of order given by

$$I_A \approx 17000\beta\gamma \text{ amperes,} \tag{1.1}$$

where β is the particle stream velocity divided by the velocity of light, and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. Qualitatively, it is easy to see how this limit comes about. The uniform current density assumption implies a magnetic field within the beam proportional to radius, and electrostatic neutralization implies that the energy is a constant. Therefore, we are able to integrate the equations of motion to obtain the particle trajectories (see Eq. (2.66)) shown in Fig. 1. (They are drawn for particles without angular momentum.) If the net current included within the maximum radial position of a particle is small compared to I_A, its motion is approximately sinusoidal, as shown by trajectory a in Fig. 1. As the included current increases, the trajectory passes through the beam axis at a greater angle (trajectory b) until at an included current of 17000βγ amperes, the particle passes through the axis perpendicular to it (trajectory c). If the included current is increased still further, net particle motion is soon backward, as shown by orbit e, and the extreme case of orbit f. Therefore, we cannot have currents in excess of about I_A under the above assumptions. It should be noted that this limit is independent of any physical dimensions. The beam current can be written

$$I \equiv Ne\beta c \equiv 17000\nu\beta, \tag{1.2}$$

where ν is the number of electrons per classical electron radius ($r_o = 2.82\text{x}10^{-15}$ meters) of beam length,

$$\nu \equiv \frac{Ne^2}{4\pi\varepsilon_o mc^2} \equiv Nr_o; \tag{1.3}$$

current in this model is limited to $\nu \lesssim \gamma$. The velocity of light is c, e is the magnitude of the electron charge, m is the electron rest mass, and ε_o is the permittivity of free space.

J. D. Lawson[3] also considered the uniform beam model in treating both partially and fully electrostatically neutralized electron beams. He arrived at a current limit of I_A for a fully neutralized beam by arguments similar to Alfvén's as well as by simply requiring that a beam electron Larmor radius in the maximum self-field of the beam of the same

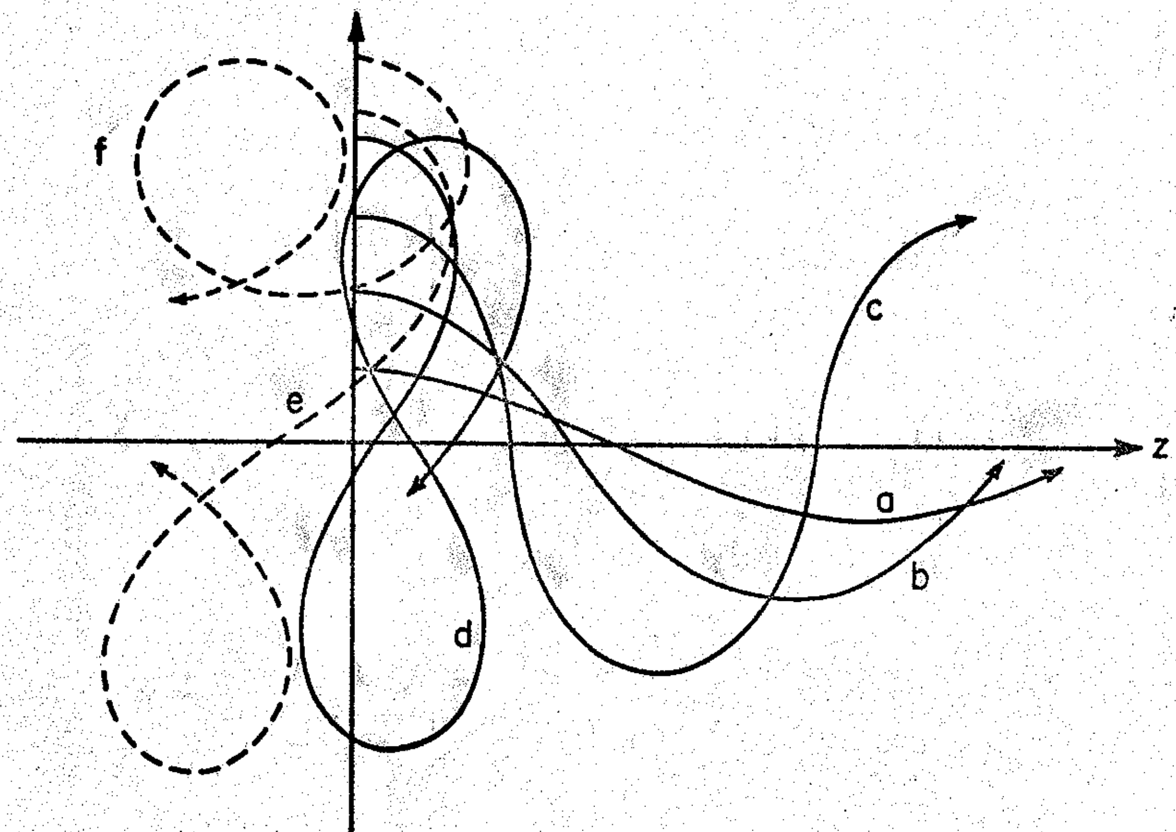

Figure 1

Trajectories of Particles Starting in the z Direction
at Various Distances from the Axis of a Uniform, Neutralized Particle Beam
(After Alfven[2])

Solid (dashed) curves represent particle trajectories with net motion forward (backward).

order as the beam radius. For an arbitrary fractional electrostatic neutralization, f, Lawson obtained a current limit of $17000\beta^3\gamma/(\beta^2 + f-1)$. In principle, then, arbitrarily large currents could be carried by a uniform beam if one carefully adjusted f to be $1-\beta^2$, or at least within the range given by

$$1-\beta^2 - \frac{\gamma\beta^2}{2\nu} > f > 1-\beta^2 + \frac{\gamma\beta^2}{2\nu} , \qquad (1.4)$$

a balancing act which is difficult to do experimentally if ν/γ is to be large compared to one.

Led by J. C. Martin and his co-workers of the United Kingdom Atomic Energy Authority, a high voltage pulse technology has recently been developed which is capable of the production of short ($\leq 10^{-7}$ second) bursts of relativistic electrons with currents in excess of I_A.[4-10] The two most striking experimental results to date are the following: 1.) At low ambient pressure ($\leq$.01 torr) in the beam drift region, Graybill, Uglam and Nablo were unable to propagate beams with more current than about 1/2 I_A.[11] 2.) At higher ambient pressures ($\geq$.1 torr), Yonas and Spence[12] and Andrews, et al.,[13] have propagated currents well over I_A.

The first observation fits in well with the current limits for neutralized beams of Alfvén and Lawson. The second result, however, led us to the development of theoretical, self-consistent beam models which allow the propagation of relativistic electron fluxes in excess of I_A. In Section II, we present and develop a simple, fully relativistic self-consistent equilibrium which can carry arbitrarily large currents when near or at complete electrostatic neutralization.

A second possible way to propagate arbitrarily large currents is if the beam is magnetically neutralized as well as electrostatically neutralized. By magnetic neutralization, we mean that large numbers of electrons from a background plasma are counterstreaming slowly within the beam so that the net current density in the system, and therefore the magnetic field, is nearly zero. There would then be no fields acting on the particles of the beam and (ignoring the obvious problem of instabilities) they would propagate in nearly straight lines. In current limit terminology, the limit would be $17000\beta^3\gamma/[\beta^2(1-f_m) - (1-f)]$, where f_m is the fractional magnetic, or current, neutralization. This mode of beam propagation has been proposed by Yonas and Spence, Andrews, et al., and others[14], as the mechanism responsible for the second experimental result given above, and some experimental verification of this has been obtained by these workers. In Section III, we solve an initial value problem for a beam-plasma system which indicates that magnetic neutralization as described above can be expected to occur for plasma densities large compared to beam densities. In Section IV, we apply a strong axial magnetic field to a uniform beam and find that it will then be able to propagate regardless of the beam current.

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II. NON-UNIFORM BEAM EQUILIBRIUM

We wish to consider here a fully relativistic equilibrium electron beam solution to the Vlasov Equation,

$$\frac{\partial f_e}{\partial t} + \underline{v}\cdot\frac{\partial f_e}{\partial \underline{x}} - e(\underline{E}+\underline{v}\times\underline{B})\cdot\frac{\partial f_e}{\partial \underline{p}} = 0, \tag{2.1}$$

and the relevant Maxwell's Equations,

$$\nabla\times\underline{B} = \mu_o\underline{j}, \tag{2.2}$$

$$\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_o} \quad . \tag{2.3}$$

The electric field and magnetic induction are $\underline{E}$ and $\underline{B}$, respectively, f_e is the electron distribution function, $\underline{v}$, $\underline{p}$ and $-e$ are the electron velocity, momentum and charge, respectively, and $\underline{j}$ and ρ are the current and charge densities. We are using MKS units so that $(\mu_o\varepsilon_o)^{-\frac{1}{2}} = c$, the free space velocity of light. The beam is infinitely long and without variation in the z direction, cylindrically symmetric, and confined to a finite radius, b. We also assume an immobile positive ion background which partially or fully neutralizes the electron beam charge density. There are no external fields.

The constants of the motion for an electron in the assumed beam are the Hamiltonian, H, the canonical axial momentum, P_z, and the angular momentum, p_θ, which are given by

$$\begin{aligned} H &= \gamma(r)mc^2 - e\Phi(r) \\ &= c[m^2c^2 + p_\perp^2 + (P_z+eA_z(r))^2]^{\frac{1}{2}} - e\Phi(r) \end{aligned} \tag{2.4}$$

$$P_z = \gamma m v_z - eA_z \equiv p_z - eA_z \tag{2.5}$$

$$p_\theta = \gamma m r^2 \omega . = \gamma m(x v_y - y v_x) \tag{2.6}$$

$\Phi(r)$ and $A_z(r)$ are the electrostatic and magnetic potentials, which are functions only of r, the radial position. The electron mass is m, ω is its angular velocity, and p_z and $p_\perp$ are the electron's parallel and perpendicular (relative to the z axis) ordinary momenta, respectively. Any function of these constants of the motion is a solution to Eq. (2.1), so we choose the particularly simple, but interesting case of mono-energetic electrons having the same axial canonical momentum,

$$f_e(\underline{x},\underline{p}) \equiv f_e(r,\underline{p}) = C_e \delta(H-\varepsilon_e)\ \delta(P_z-\gamma_o m V_z) \quad . \tag{2.7}$$

Defining $A_z(r=0) = 0 = \Phi(r=0)$, we have from Eqs. (2.4) and (2.5) that γ_o and V_z are the values of γ and v_z for an electron at $r = 0$, and that $\varepsilon_e = \gamma_o mc^2$. C_e is a normalization constant to be determined. The first two moments of this distribution function are

$$n_e(\underline{x}) \equiv n_e(r) = \int d\underline{p}\ f_e(r,\underline{p}) \equiv \int_{-\infty}^{\infty} dp_z \int_0^{\infty} p_\perp dp_\perp \int_0^{2\pi} d\theta\ f_e(r,\underline{p}) \tag{2.8}$$

$$n_e\langle v_z\rangle = \int_{-\infty}^{\infty} dp_z \int_0^{\infty} p_\perp dp_\perp \int_0^{2\pi} d\theta\ f_e(r,\underline{p})\ \frac{p_z}{\gamma m} \quad . \tag{2.9}$$

(Since f_e and γ are even functions of p_x, $n_e\langle v_x\rangle = 0$ and $n_e\langle v_y\rangle = 0$.) Because of the δ function in P_z, we can write

$$\gamma = \left(1 + \frac{(\gamma_o mV_z+eA_z)^2}{m^2c^2} + \frac{p_\perp^2}{m^2c^2}\right)^{\frac{1}{2}} \equiv \left(\alpha^2 + \frac{p_\perp^2}{m^2c^2}\right)^{\frac{1}{2}} \equiv \alpha u \quad , \tag{2.10}$$

where

$$\alpha^2 \equiv 1 + \frac{(\gamma_o mV_z+eA_z)^2}{m^2c^2} \quad , \tag{2.11}$$

$$u \equiv \left(1 + \frac{p_\perp{}^2}{m^2c^2\alpha^2}\right)^{\frac{1}{2}} \quad , \tag{2.12}$$

and we change the variable of integration from $p_\perp$ to u. Performing the θ and p_z integrations, therefore, gives

$$n_e(r) = 2\pi C_e \int_1^\infty udu(mc\alpha)^2 \; \delta(mc^2\alpha u - e\Phi - \varepsilon_e) \tag{2.13a}$$

$$n_e\langle v_z\rangle = 2\pi C_e \int_1^\infty udu\frac{\gamma_o mV_z+eA_z}{\alpha um} \; (mc\alpha)^2 \; \delta(mc^2\alpha u - e\Phi - \varepsilon_e) \quad . \tag{2.13b}$$

The δ-functions in these integrations are zero over the whole range of the u-integration unless $e\Phi + \varepsilon_e \geq mc^2\alpha$. Thus, we have

$$n_e(r) = \begin{cases} \dfrac{2\pi C_e(e\Phi+\varepsilon_e)}{c^2} & r \leq b \\[2ex] 0 & r > b \end{cases} \tag{2.14}$$

where b is defined by

$$\frac{\varepsilon_e+e\Phi(b)}{mc^2\alpha(b)} = 1 \tag{2.15}$$

and is the beam radius. (We have assumed that the δ function argument has at most one zero over the interval of integration, which checks out later.) Since $\Phi(0) = 0$, $C_e = n_e(0)c^2/2\pi\varepsilon_e$, and we obtain

$$n_e(r) = \begin{cases} n_e(0)\left(1 + \dfrac{e\,\Phi(r)}{\varepsilon_e}\right) & r \le b \\ 0 & r > b \end{cases} \tag{2.16}$$

$$n_e\langle v_z\rangle = \begin{cases} \dfrac{n_e(0)c^2}{\varepsilon_e}\,[\gamma_o m V_z + eA_z(r)] & r \le b \\ 0 & r > b \end{cases} \tag{2.17}$$

We can now obtain the self-fields of the beam assuming that the background ions provide a charge neutralization fraction f, $0 \le f \le 1$. The potential equations obtained from Eqs. (2.2) and (2.3) are

$$\nabla\cdot\underline{E} \equiv -\nabla^2\Phi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\Phi}{\partial r}\right) = \begin{cases} \dfrac{e(f-1)}{\varepsilon_o}\,n_e(0)\left[1+\dfrac{e\Phi}{\varepsilon_e}\right] & r \le b \\ 0 & r > b \end{cases} \tag{2.18}$$

$$(\nabla\times\underline{B})_z \equiv [\nabla\times(\nabla\times\underline{A})]_z = -\nabla^2 A_z = \begin{cases} \dfrac{n_e(0)e}{\varepsilon_o\varepsilon_e}\,(\gamma_o m V_z + eA_z) & r \le b \\ 0 & r > b \end{cases} \tag{2.19}$$

Let

$$\frac{1}{L_e^2} = \frac{n_e(0)e^2}{\varepsilon_o \varepsilon_e} \tag{2.20a}$$

$$\Phi_o = -\frac{n_e(0)L_e^2 e}{\varepsilon_o} = -\frac{\varepsilon_e}{e} \tag{2.20b}$$

$$A_o = -\frac{n_e(0)\gamma_o m V_z e L_e^2}{\varepsilon_o \varepsilon_e} = -\frac{\gamma_o m V_z}{e} \quad . \tag{2.20c}$$

Then Eqs. (2.18) and (2.19) for $r \leq b$ assume the form of modified Bessel equations of order zero for dependent variables $\Phi-\Phi_o$ and A_z-A_o. Taking the solution for which $\Phi(0) = 0 = A_z(0)$ there results

$$\Phi(r) = \begin{cases} \Phi_o \left\{1 - I_o\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right)\right\} & r \leq b \\ \Phi_o \left\{1 - I_o\left(\frac{b}{L_e}(1-f)^{\frac{1}{2}}\right)\right\} - \frac{b}{L_e}(1-f)^{\frac{1}{2}}\, \Phi_o I_1\left(\frac{b}{L_e}(1-f)^{\frac{1}{2}}\right) \ln\frac{r}{b} & r > b \end{cases} \tag{2.21}$$

$$A_z(r) = \begin{cases} A_o \left\{1 - I_o\left(\frac{r}{L_e}\right)\right\} & r \leq b \\ A_o \left\{1 - I_o\left(\frac{b}{L_e}\right)\right\} - \frac{b}{L_e} A_o I_1\left(\frac{b}{L_e}\right) \ln\frac{r}{b} & r > b \end{cases} \tag{2.22}$$

Where I_m is the modified Bessel function of the first kind and order m. The solutions for r > b were obtained by integrating from r = b. These give for the electric and magnetic fields,

$$E_r(r) = \begin{cases} \Phi_o \dfrac{(1-f)^{\frac{1}{2}}}{L_e} I_1\left(\dfrac{r}{L_e}(1-f)^{\frac{1}{2}}\right) & r \leq b \\ \Phi_o \dfrac{b(1-f)^{\frac{1}{2}}}{rL_e} I_1\left(\dfrac{b}{L_e}(1-f)^{\frac{1}{2}}\right) & r > b \end{cases} \tag{2.23}$$

$$B_\theta(r) = \begin{cases} \dfrac{A_o}{L_e} I_1\left(\dfrac{r}{L_e}\right) & r \leq b \\ \dfrac{A_o b}{rL_e} I_1\left(\dfrac{b}{L_e}\right) & r > b. \end{cases} \tag{2.24}$$

If we substitute Φ for $r \leq b$ into Eqs. (2.4) and (2.16), we obtain

$$\frac{n_e(r)}{n_e(o)} = I_o\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right) = \frac{\gamma(r)}{\gamma_o}. \tag{2.25}$$

Then using A_z from Eq. (2.22) and $n_e(r)$ from (2.25) in Eq. (2.17) gives

$$v_z(r) = V_z \frac{I_o\left(\frac{r}{L_e}\right)}{I_o\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right)} . \tag{2.26}$$

Thus, since $I_o(0) = 1$, if the beam is neutralized (f = 1) the density is uniform, as is γ. However, the axial velocity distribution (and therefore j_z) are far from uniform for $\frac{r}{L_e} >> 1$, because[15]

$$I_n(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}} \left(1 - \frac{4n^2-1}{8x} + \ldots\right) \qquad \text{for } x >> 1. \tag{2.27}$$

Since $v_z(b)$ is limited by c, this means that $V_z << c$.

The circumstance under which $\frac{r}{L_e} >> 1$ is that the total current, I, being carried by the beam be large compared to I_A, as we shall now show. From (2.24) and Ampere's Law,

$$I = \frac{1}{\mu_0} \oint \underline{B} \cdot \underline{dl} \Big|_{r=b} = - 2\pi \frac{\varepsilon_o mc^3}{e} \gamma_o \beta_z \frac{b}{L_e} I_1\left(\frac{b}{L_e}\right) . \tag{2.28}$$

Defining I_A for this non-uniform beam analogously to one of Lawson's derivations of it,[3] we take I_A as the current for which a beam electron Larmor radius, R_L, is half the beam radius:

$$R_L = \frac{\gamma(b) m v_z(b)}{e B_\theta(b)} = \frac{b}{2} . \tag{2.29}$$

Using Eqs. (2.24) - (2.26), we readily obtain

$$I_A = - \frac{4\pi\varepsilon_o mc^3}{e} \gamma_o \beta_z(b) \approx -17000 \beta_z(b) \gamma_o, \tag{2.30}$$

where $\beta_z(b) = v_z(b)/c$. Therefore

$$\frac{I}{I_A} = \frac{1}{2} \frac{b}{L_e} \frac{I_1\left(\frac{b}{L_e}\right)}{I_o\left(\frac{b}{L_e}\right)} \sim \frac{1}{2} \frac{b}{L_e} - \frac{1}{4} , \tag{2.31}$$

the asymptotic form coming for $\frac{b}{L_e} >> 1$. This says that arbitrarily large current can be carried within a given radius, b, so long as the sheath thickness, L_e, is sufficiently small compared to b -- so long as the beam can be created with sufficiently high density. Note that L_e is the usual collisionless skin depth, $\frac{c}{\omega_{p_b}}$, where

$$\omega_{p_b}^2 = \frac{n_e(o)e^2}{\gamma_o \varepsilon_o m} \qquad (2.32)$$

is the electron plasma frequency of the beam for all r (on account of Eq. (2.25)). Hence, if $I >> I_A$, E_r and B_θ drop off nearly exponentially inside of r = b, becoming small compared to their maxima inside the depth $\frac{c}{\omega_{p_b}}$ from r = b. This means that particles which start out at r = b with $v_z = V_z I_o\left(\frac{b}{L_e}\right)$ (for f = 1), leave the region of high magnetic field before they have had a chance to turn around, as they could in the "uniform beam" case, Fig. 1.

Even though all of the above equations are valid for any f we find that it is not possible for the equilibrium to exist unless there is a certain minimum of neutralization. If we restrict the maximum energy that an electron can gain in the electric field to $(\gamma(b)-1)mc^2$, then we may write

$$\gamma_o = \frac{\gamma(b)}{I_o\left(\frac{b}{L_e}(1-f)^{\frac{1}{2}}\right)} \geq 1 \; . \qquad (2.33)$$

This implies that

$$f \geq 1 - \{\frac{L_e}{b} I_o^{-1}[\gamma(b)]\}^2 \quad , \tag{2.34}$$

where the symbol $I_o^{-1}[\gamma(b)]$ means the argument, χ, for which $I_o(\chi) = \gamma(b)$. Therefore, unless $\gamma(b)$ is unlimited, if $\frac{b}{L_e} >> 1$, f will be limited to very near 1. For example, if $\gamma(b) = 6$ (≈ 2.5 megavolts of kinetic energy) and I = 200,000 amperes, then at minimum f ($\gamma_o = 1$), $\frac{I}{I_A} \approx 12$, corresponding to $\frac{b}{L_e} = 24.5$, and a minimum f of about 0.98! If we wish to put no limit on $\gamma(b)$ the consequences are impractical -- f = .5 and $\gamma_o = 1$ for $\frac{b}{L_e} = 24.5$ gives $\gamma(b) \approx 2\times10^6$. Thus, we are reasonably justified in sticking to f = 1 in most of what follows.

Since f = 1 implies a uniform electron density within the beam the quantity ν defined in Eq. (1.3) is again useful. Here it is

$$\nu = \frac{n_e(o)\pi b^2 e^2}{4\pi\varepsilon_o mc^2} = \frac{\gamma_o b^2}{4L_e^2} \quad . \tag{2.35}$$

This implies that

$$(\frac{\nu}{\gamma_o})^{\frac{1}{2}} = \frac{b}{2L_e} \approx \frac{I}{I_A} \tag{2.36}$$

for a high current beam. Since I, γ_o and b are usually experimentally measureable, ν, ω_{p_b} and c/ω_{p_b} could be calculated and compared with other measurements, such as density measurements, or characteristic lengths for fields. Another experimentally measureable quantity is the propagation velocity for the bulk of an electron beam[12,13].

For f = 1, I ≡ $17000\nu\bar{\beta}$ can be used to define such an average velocity $\bar{\beta}c$ if the current is known. Using (2.36) and (2.30), together with $\beta_z^2(b)\gamma_o^2 = \gamma_o^2 - 1$, we obtain for $I >> I_A$

$$\bar{\beta} \approx \frac{17000}{I} \frac{\gamma_o^2 - 1}{\gamma_o} \quad . \tag{2.37}$$

Thus, if $I \approx 10^5$ amperes and $\gamma_o \approx 2$, $\bar{\beta} \approx \frac{1}{4}$, compared with $\beta_z(b) \approx .86$. Finally, for f = 1,

$$\gamma_o^2 = \frac{1}{1-\beta_\perp^2-\beta_z^2} = \frac{1}{1-\beta_z^2 I_o^2\left(\frac{b}{L_e}\right)} \quad , \tag{2.38}$$

where $\beta_\perp c$ is the radial velocity of an electron at r = 0. Consequently

$$\beta_\perp^2 = \beta_z^2[I_o^2\left(\frac{b}{L_e}\right) - 1], \tag{2.39}$$

so that for a high current beam, $\beta_\perp^2 >> \beta_z^2$.

So far we have learned a great deal about this equilibrium without knowing anything about the details of electron motion. And it is clear that the self-consistent fields given by Eqs. (2.23) and (2.24) are such that electron orbits will not easily be obtained from the equations of motion. In fact, however, it is possible to obtain an orbit integral for quite a general equilibrium $\left(\frac{\partial}{\partial t} = 0\right)$, infinite cylindrically symmetric $\left(\frac{\partial}{\partial \phi} = 0\right)$ beam with no axial variation $\left(\frac{\partial}{\partial z} = 0\right)$ using the three constants of the motion given by Eqs. (2.4) - (2.6). We may even have an axial magnetic field, via a vector potential component A_Θ, if it is a function only of r. In this case Eq. (2.6) becomes

$$p_\theta = \gamma m r^2 \omega - eA_\theta(r)r \quad . \tag{2.40}$$

We merely have to solve

$$\gamma mc^2 = c[m^2c^2 + p_r{}^2 + \left(\frac{p_\theta}{r} + eA_\theta(r)\right)^2 + \left(p_z + eA_z(r)\right)^2]^{\frac{1}{2}} \tag{2.41}$$

for the radial momentum, p_r. If a subscript a implies the quantity evaluated at some initial time, t_a, we obtain

$$\frac{dr}{dt} = \pm \frac{1}{\{1 - \frac{e}{\gamma_a mc^2}\left(\phi_a - \phi(r)\right)\}} \left\{ v_{ra}{}^2 + \frac{r_a{}^2 \omega_a{}^2}{r^2}(r^2 - r_a{}^2) \right.$$

$$+ \frac{2e}{\gamma_a m}[\left(A_\theta(r) - \frac{r_a}{r} A_{\theta a}\right) \frac{r_a{}^2 \omega_a}{r} + v_{za}\left(A_{za} - A_z(r)\right) - \left(\phi_a - \phi(r)\right)]$$

$$\left. - \left(\frac{e}{\gamma_a m}\right)^2 [\left(A_\theta(r) - \frac{r_a}{r} A_{\theta a}\right)^2 + \left(A_{za} - A_z(r)\right)^2 - \frac{1}{c^2}\left(\phi_a - \phi(r)\right)^2] \right\}^{\frac{1}{2}}$$

$$\equiv \pm g(r). \tag{2.42}$$

We have taken $p_r \equiv \gamma m v_r \equiv \gamma m \frac{dr}{dt}$ in this equation. We therefore obtain the quadrature

$$t(r) - t_a = \pm\int_{r_a}^{r} \frac{dr'}{g(r')} \quad . \tag{2.43}$$

The turning points of the radial motion are, of course, the zeros of $\pm g(r) \equiv \frac{dr}{dt}$, so that Eq. (2.43) is defined only if r is between these turning points. The sign to be taken is + (-) according to whether r is greater (less) than r_a. The potentials A_z and Φ must, of course, be the self-consistent ones for the beam. For the present beam, using Φ and A_z from Eqs. (2.21) and (2.22), $A_\Theta \equiv 0$, and eliminating γ_a and v_{za} in favor of γ_o and V_z from Eqs. (2.25) and (2.26), we obtain

$$g(r) = \frac{I_o\left[\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right]}{I_o\left[\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right]} \left\{ v_{ra}{}^2 + \frac{r_a{}^2\omega_a{}^2}{r^2}(r^2 - r_a{}^2) + \frac{c^2}{I_o{}^2\left[\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right]} \right.$$

$$\left. \times \left[I_o{}^2\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right) - I_o{}^2\left(\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right) + \beta_z{}^2\left[I_o{}^2\left(\frac{r_a}{L_e}\right) - I_o{}^2\left(\frac{r}{L_e}\right)\right]\right] \right\}^{\frac{1}{2}} . \tag{2.44}$$

This form exhibits explicitly that r_a is a radial turning point if $v_{ra} = 0$. However a somewhat simpler form is obtained by making the substitutions

$$\frac{1}{\gamma_a{}^2} = \frac{1}{\gamma_o{}^2 I_o{}^2\left(\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right)} = 1 - \beta_z{}^2 \frac{I_o{}^2\left(\frac{r_a}{L_e}\right)}{I_o{}^2\left(\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right)} - \frac{r_a{}^2\omega_a{}^2}{c^2} - \frac{v_{ra}{}^2}{c^2} \tag{2.45}$$

and

$$(r_a{}^2\omega_a)^2 = \frac{p_\theta{}^2}{(\gamma_a m)^2} = \frac{p_\theta{}^2}{(\gamma_o m)^2 I_o{}^2\left(\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right)} \quad . \tag{2.46}$$

The result is

$$t(r)-t_a = \pm\int_{r_a}^{r} \frac{r'dr' \; I_o\left(\frac{r'}{L_e}(1-f)^{\frac{1}{2}}\right)}{\{c^2 r'^2 [I_o{}^2\left(\frac{r'}{L_e}(1-f)^{\frac{1}{2}}\right) - \frac{1}{\gamma_o{}^2} - \beta_z{}^2 I_o{}^2\left(\frac{r'}{L_e}\right)] - \frac{p_\theta{}^2}{(\gamma_o m)}\}^{\frac{1}{2}}} \tag{2.47}$$

The remaining components of the motion can be put in integral form similarly:

$$z(r)-z_a = \int_{t_a}^{t} v_z(t')dt' = \int_{r_a}^{r} v_z(r')\frac{dt}{dr'}\,dr' \tag{2.48}$$

$$\theta(r)-\theta_a = \int_{t_a}^{t} \omega(t')dt' = \int_{r_a}^{r} \omega(r')\,\frac{dt}{dr'}\,dr' \quad . \tag{2.49}$$

$$\omega(r') = \frac{I_o\left(\frac{r_a}{L_e}(1-f)^{\frac{1}{2}}\right)}{I_o\left(\frac{r'}{L_e}(1-f)^{\frac{1}{2}}\right)} \frac{r_a{}^2\omega_a}{r'^2} \tag{2.50}$$

is readily obtained from the constancy of p_θ together with Eq. (2.25). Hence, formally, we have the electron orbits for all allowed values of p_θ. Useful results, of course, require numerical computation or approximations.

We can also formally determine the distribution of angular momentum, $F(p_\theta)$, for the electrons in this beam model. By definition

$$F(p_\theta) = \int d\underline{x}\, d\underline{p}\, f_e\, (\underline{x}, \underline{p})\, \delta[p_\theta - (xp_y - yp_x)]. \tag{2.51}$$

With the substitutions $x = r\cos\phi$, $y = r\sin\phi$, $p_x = p_\perp\cos\theta$ and $p_y = p_\perp\sin\theta$, Eq. (2.51) is

$$F(p_\theta) = \frac{n_e(o)c^2}{2\pi\varepsilon_e} \int dz\; rdrd\phi\; dp_z p_\perp dp_\perp\, d\Theta\; \delta[p_z - (\gamma_o mV_z + eA_z)]$$

$$\times\; \delta[p_\theta - p_\perp r\sin(\theta-\phi)]\delta[\gamma mc^2 - e\Phi - \varepsilon_e] \quad . \tag{2.52}$$

Using

$$\int_0^{2\pi} d\theta\delta[p_\theta - p_\perp r\sin(\theta-\phi)] = \frac{2}{(p_\perp^2 r^2 - p_\theta^2)^{\frac{1}{2}}} \quad , \tag{2.53}$$

we then have simply $\int dzd\phi \equiv 2\pi$, making $F(p_\theta)$ the distribution of angular momentum per unit length of beam. The α and u substitutions of Eqs. (2.11) and (2.12) then enable us to obtain

$$F(p_\theta) = \frac{2n_e(0)}{\gamma_o m} \int_{r_1}^{r_2} \frac{rdr\; I_o\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right)}{\left\{r^2c^2\left[I_o^{\,2}\left(\frac{r}{L_e}(1-f)^{\frac{1}{2}}\right) - \frac{1}{\gamma_o^{\,2}} - \beta_z^{\,2} I_o^{\,2}\left(\frac{r}{L_e}\right)\right] - \frac{p_\theta^2}{(\gamma_o m)^2}\right\}^{\frac{1}{2}}} \,. \tag{2.54}$$

r_1 and r_2 are the inner and outer turning points of a particle with angular momentum p_θ. From this we can see that $p_\theta^{\,2}$ can be anything from 0 to $p^2_{\theta max}$, the maximum of the function

$$h(r) \equiv (\gamma_o mcr)^2 [I_o^2(\frac{r}{L_e}(1-f)^{\frac{1}{2}}) - \frac{1}{\gamma_o^2} - \beta_z^2 I_o^2(\frac{r}{L_e})]. \qquad (2.55)$$

For large currents and f=1 it is reasonable to obtain the radius, R, at which a particle with $p_{\theta max}$ circulates. By writing $\frac{\partial h(r)}{\partial r} = 0$ (or by equating the centripetal force to the Lorentz force for a particle with only v_θ and v_z) we obtain

$$1 - \frac{1}{\gamma_o^2} - \beta_z^2 I_o^2(\frac{R}{L_e}) = \beta_z^2 \frac{R}{L_e} I_o(\frac{R}{L_e}) I_1(\frac{R}{L_e}) . \qquad (2.56)$$

Using $\beta_z^2 = \beta_z^2(b)/I_o^2(\frac{b}{L_e}) = (1 - \frac{1}{\gamma_o^2})/I_o^2(\frac{b}{L_e})$, the asymptotic expansions for I_o and I_1 (Eq. (2.27)) give

$$\frac{R}{L_e} \approx \frac{1}{2}\frac{b}{L_e} - \frac{1}{4}\ln\frac{b}{L_e} + \frac{1}{2}\left(\frac{1}{4}\left(\frac{2b}{L_e} - \ln\frac{b}{L_e}\right)^2 - \frac{3}{2}\right)^{\frac{1}{2}} \approx \frac{b}{L_e} - \frac{1}{2}\ln\frac{b}{L_e} . \qquad (2.57)$$

Hence, for $I >> I_A$, the particle with $p_{\theta max}$ circulates rather far out compared with the uniform beam result which we will find to be $b/(2)^{\frac{1}{2}}$. For example, if $\gamma_o = 2$ and $I = 10^5$ amperes, then $I/I_A = 3.4$ so that $\frac{b}{L_e} = 7.3$, and $\frac{R}{L_e} \approx 6.3$. In this case, then, a particle with $p_{\theta max}$ will have a (constant) z velocity of about $\frac{1}{3}$ c, compared with $\beta(b) \approx .865$ and $\bar{\beta} \approx \frac{1}{4}$ found above.

Returning now to $F(p_\theta)$, comparison of Eqs. (2.47) and (2.54) reveals that if $\tau(p_\theta)$ is the time it takes for a particle with angular momentum p_θ to go from its inner to its outer turning point - r_1 to r_2 - then

$$F(p_\theta) = \frac{2n_e(0)}{\gamma_o m} \tau(p_\theta). \tag{2.58}$$

The current being carried by particles with angular momentum between p_θ and $p_\theta + dp_\theta$ may be written

$$dI(p_\theta) = - eF(p_\theta)\, \bar{v}_z(p_\theta) dp_\theta, \tag{2.59a}$$

where $\bar{v}_z(p_\theta)$ is the average z velocity of a particle with angular momentum, p_θ. Defining $Z(p_\theta)$ as the z distance such a particle travels between turning points, Eq. (2.58) becomes

$$\frac{dI}{dp_\theta} = - \frac{2n_e(0)}{\gamma_o m} Z(p_\theta). \tag{2.59b}$$

From Eq. (2.48), $Z(p_\theta)$ is given by

$$Z(p_\theta) = V_z \int_{r_1}^{r_2} \frac{r\, I_o\left(\frac{r}{L_e}\right) dr}{\left\{c^2 r^2 [I_o{}^2\left(\frac{r}{L_e}\right. (1-f)^{\frac{1}{2}}\left.\right) - \frac{1}{\gamma_o{}^2} - \beta_z{}^2 I_o{}^2\left(\frac{r}{L_e}\right)] - \frac{p_\theta{}^2}{\gamma_o{}^2 m^2}\right\}^{\frac{1}{2}}} . \tag{2.60}$$

For a low current beam, we will see that $Z(p_\theta)$ is a constant. However, for $I \gg I_A$, it is apparent from a numerical computation to follow that the higher p_θ particles contribute more to the current (have a greater $Z(p_\theta)$) than do the low p_θ particles (see Fig. 3a).

So far we have mainly discussed $I \gg I_A$. We now take a look at the low current limit. This requires that $b \ll L_e$. Since $I_o(x) \approx 1 + (\frac{1}{2}x)^2$ and $I_1(x) \approx \frac{1}{2} x$ for small x, it follows that

$$n_e(r) \simeq n_e(0)\ [1 + \frac{r^2(1-f)}{4L_e^{\ 2}}] \simeq n_e(0) \qquad (2.61a)$$

$$\gamma(r) \simeq \gamma_o[1 + \frac{r^2(1-f)}{4L_e^{\ 2}}] \simeq \gamma_o \qquad (2.61b)$$

$$v_z(r) \simeq V_z[1 + \frac{r^2}{4L_e^{\ 2}} f] \simeq V_z \qquad (2.61c)$$

$$\Phi(r) \simeq \frac{n_e(0)e(1-f)r^2}{4\varepsilon_o} \qquad r \leq b \qquad (2.61d)$$

$$A_z(r) \simeq \frac{n_e(0)eV_z r^2}{4\varepsilon_o c^2} \qquad r \leq b \qquad (2.61e)$$

$$E_r \simeq -\frac{n_e(0)e(1-f)r}{2\varepsilon_o} \qquad r \leq b \qquad (2.61f)$$

$$B_\theta \simeq -\frac{n_e(0)eV_z r}{2\varepsilon_o c^2} \qquad r \leq b \qquad (2.61g)$$

All of these are characteristic of the uniform beam. Consequently, it emerges as the self consistent, fully relativistic solution to the Vlasov Equation for a low current beam - $I << I_A$. This was obtained by Mjolsness[16], and non-relativistically by Longmire[17] with some small non-uniform effects. For f = 1, Eq. (2.56) reveals that in the uniform beam, the particle with $p_{\theta max}^{\ 2}$ circulates at $b/(2)^{\frac{1}{2}}$. In the case of uniform v_z and γ, and f=1, the perpendicular energy available from A_z at radius r gives

$$\nu mc^2 \beta_z^{\ 2}(1 - \frac{r^2}{b^2}) = \frac{p_r^{\ 2}}{2\gamma_o m} + \frac{p_\theta^{\ 2}}{2\gamma_o mr^2}\ . \qquad (2.62)$$

Hence,

$$p_{\theta max}^{2} = \frac{\nu\gamma_o m^2 c^2 b^2 \beta_z^{\,2}}{2} \quad . \tag{2.63}$$

and for all p_θ^2 up to $p_{\theta max}^2$, r_1 and r_2 are given by

$$r_{2,1}^{2} = \frac{b^2}{2}\left[1 \pm \left(1 - \frac{p_\theta^{\,2}}{p_{\theta max}^{2}}\right)^{\frac{1}{2}}\right] \quad . \tag{2.64}$$

Solving (2.62) for $p_r^2 \equiv \gamma_o^{\,2} m^2 \left(\frac{dr}{dt}\right)^2$ easily gives the orbit integral

$$t(r) = -\frac{b}{\left(2\frac{\nu}{\gamma_o}\right)^{\frac{1}{2}} V_z} \int_{r_2}^{r} \frac{r\,dr}{(r^2 - r_1^{\,2})^{\frac{1}{2}}(r_2^{\,2} - r^2)^{\frac{1}{2}}} \quad . \tag{2.65}$$

This gives an arcsin, and the orbits obtained by Lawson[3] and others[16,18] result. We cannot, however, obtain the trajectories of Fig. 1 from Eq. (2.65) unless we use (2.5) and allow v_z to change regardless of $\frac{\nu}{\gamma_o}$ according to

$$v_z(r) = V_z\left[1 - \frac{\nu}{\gamma_o}\left(1 - \frac{r^2}{b^2}\right)\right] \quad . \tag{2.66}$$

In this equation, V_z is the maximum value of v_z, rather than the value at r=0. Under the assumptions for which (2.65) is valid, we obtain for $|p_\theta| < p_{\theta max}$

$$\tau(p_\theta) = \frac{\pi b}{2\left(2\frac{\nu}{\gamma_o}\right)^{\frac{1}{2}} V_z} \quad , \tag{2.67a}$$

$$Z(p_\theta) = \frac{\pi b}{2\left(2\frac{\nu}{\gamma_o}\right)^{\frac{1}{2}}} \quad , \tag{2.67b}$$

and

$$F(p_\theta) = \begin{cases} \frac{N}{2p_{\theta max}}, & |p_\theta| < p_{\theta max} \\ 0 & \text{otherwise,} \end{cases} \quad (2.67c)$$

and the beam is indeed, uniform.

Let us now return to high current beams and look at some numerical results. In Fig. 2 we plot h(r) for beam parameters appropriate to the beam of Andrews, et al., $\gamma_o = 2$, $I = 10^5$ amperes, and f = 1. We observe that p_θ^2 can be anything from 0 to about $101(mcL_e)^2\gamma_o/4\nu$. For any allowed p_θ, r_1 and r_2 can be obtained from the graph. A particle with $p_{\theta max}$ circulates at $\frac{R}{L_e} \simeq 6.27$, from the numerical work, compared to 6.3 found from Eq. (2.57). In Fig. 3 we plot z(r), r(t) and z(t) for several p_θ values, and find that the higher angular momentum particles go somewhat further in the z direction between radial turning points. From Eq. (2.59), they therefore contribute more to the current. Note, however, from Fig. 3b and Eq. (2.58), that there are more low angular momentum particles.

We now suppose that this beam could be set up with f = 1 in a "drift tube" with a perfectly conducting wall at $a \geq b$. Then the sum of the magnetic field energy per meter of beam inside and outside the beam, U, is

$$U = \gamma_o^2 K\beta_z^2 \frac{b}{L_e} [I_1(\frac{b}{L_e}) I_o(\frac{b}{L_e}) - \frac{b}{2L_e}\left\{I_o(\frac{b}{L_e}) - I_1^2(\frac{b}{L_e})\right\}]$$

$$+ \gamma_o^2 K(\frac{b}{L_e})^2 \beta_z^2 I_1^2(\frac{b}{L_e}) \ln\frac{a}{b} . \quad (2.68)$$

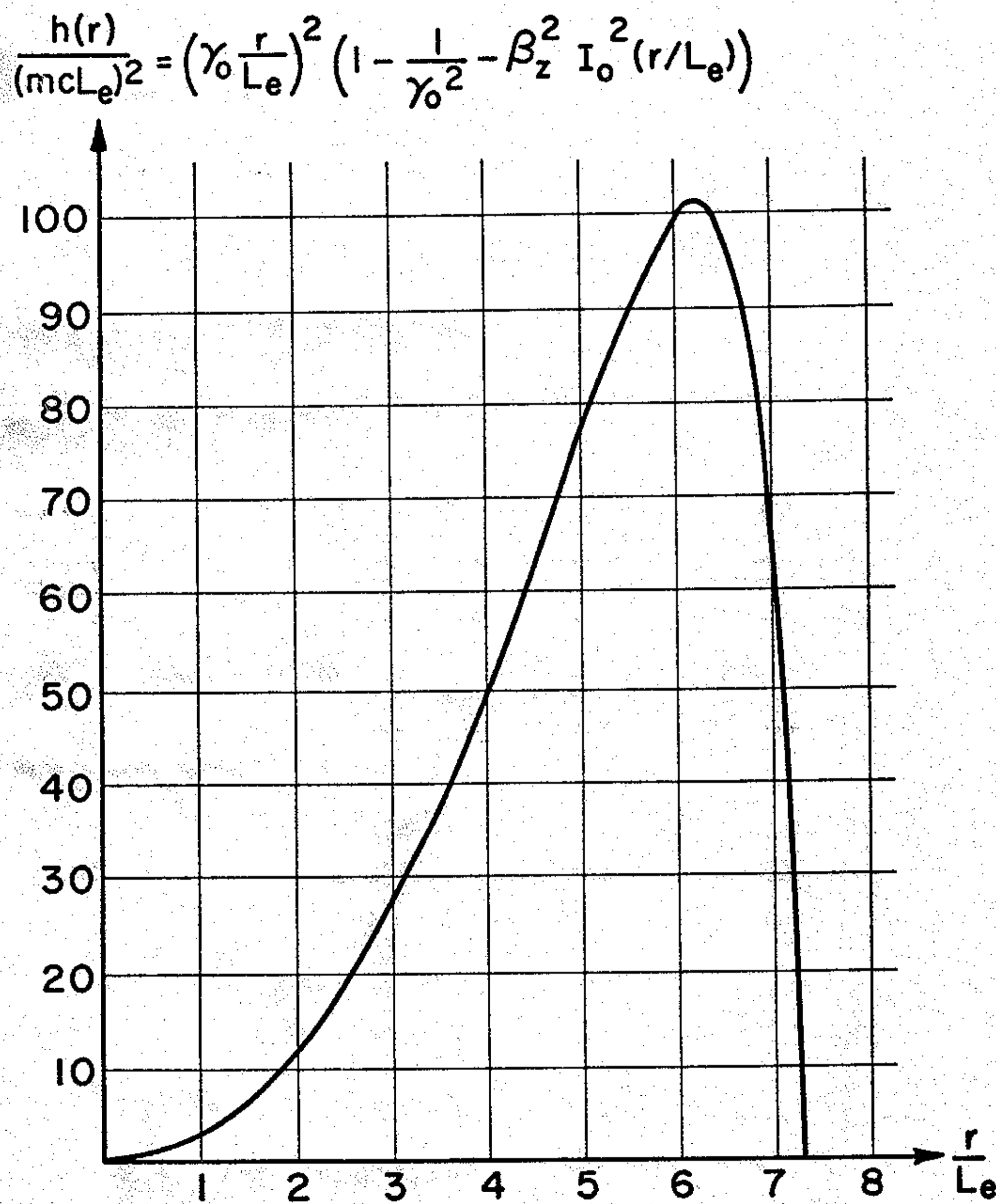


Figure 2

$h(r)$ for $f = 1$, $\gamma_0 = 2$, $I = 10^5$ A
($\beta_z = .00386$, $b/L_e = 7.3$)

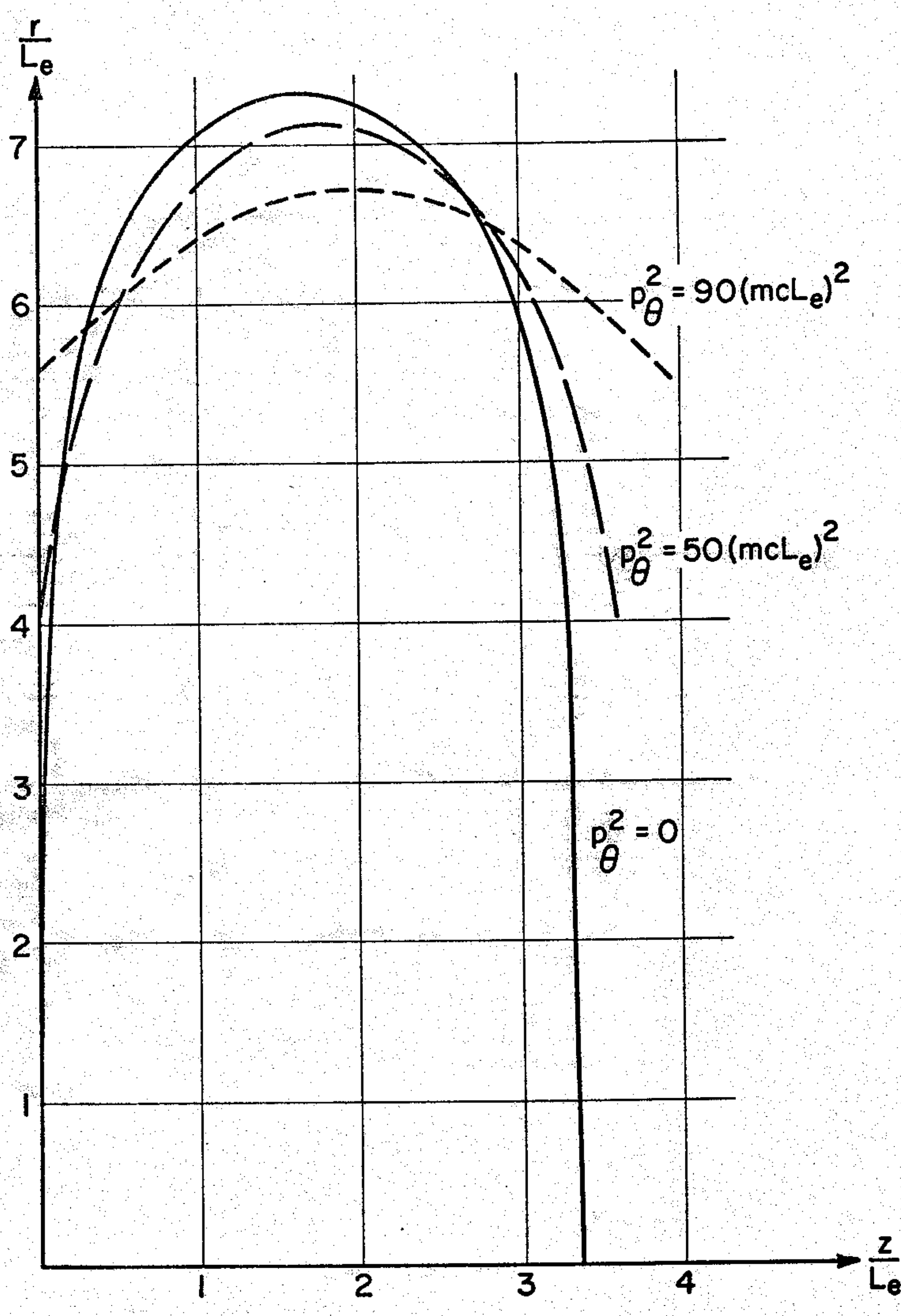


Figure 3a - Particle Radial Position Vs Axial Distance Travelled by Electrons Having $p_\theta^2 = 0$, $50(mcL_e)^2$, and $90(mcL_e)^2$. (Plot runs from r_1 to r_2 and back.)

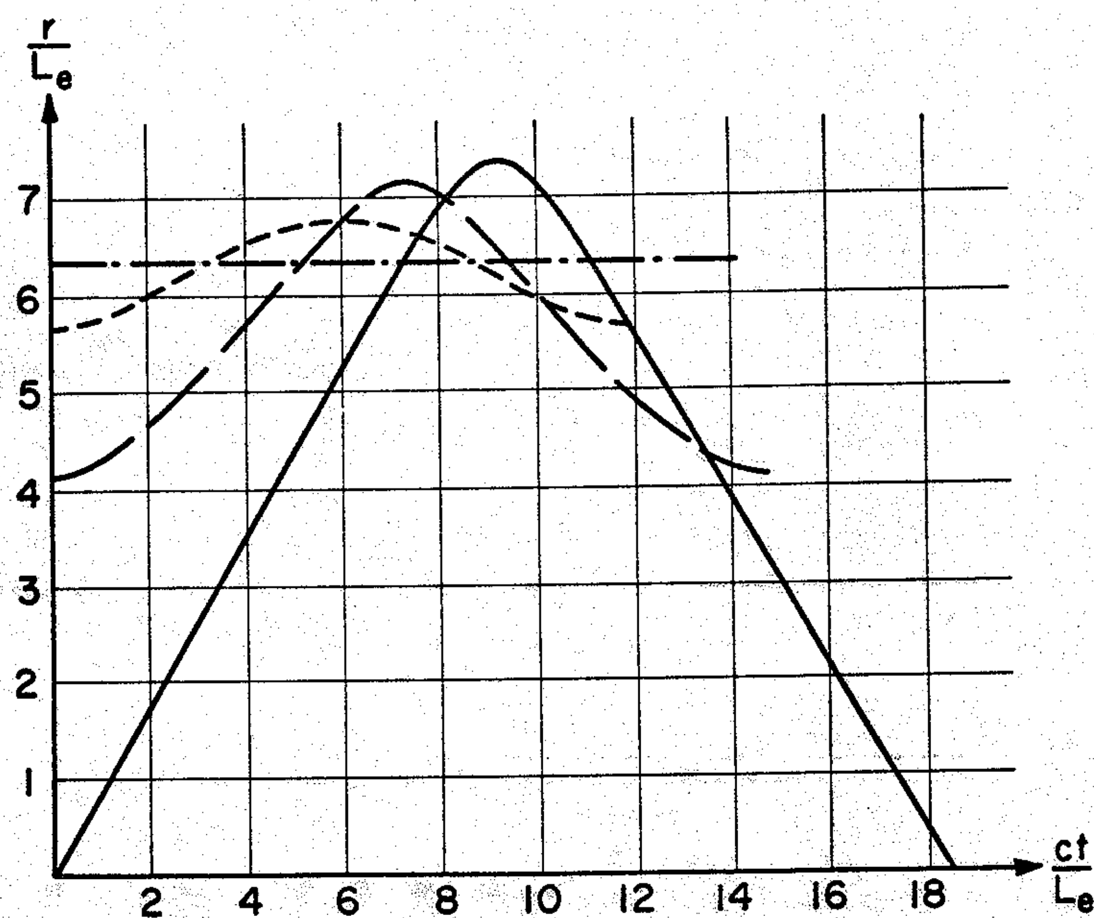


Figure 3b

$\frac{r}{L_e}$ Vs $\frac{ct}{L_e}$ for Electrons with $p_\theta^2 = 0$,
$50(mcL_e)^2$, $90(mcL_e)^2$, and $p_{\theta max}^2 \approx 101(mcL_e)^2$.

————————	$p_\theta^2 = 0$ (Plotted from $r = 0$ to $r = b$ and back to $r = 0$)
——— ———	$p_\theta^2 = 50(mcL_e)^2$ } (Plotted from r_1 to r_2 and back to r_1)
- - - - - - -	$p_\theta^2 = 90(mcL_e)^2$ } (Plotted from r_1 to r_2 and back to r_1)
——— · ———	$p_\theta^2 = p_{\theta max}^2$ (r is a constant)

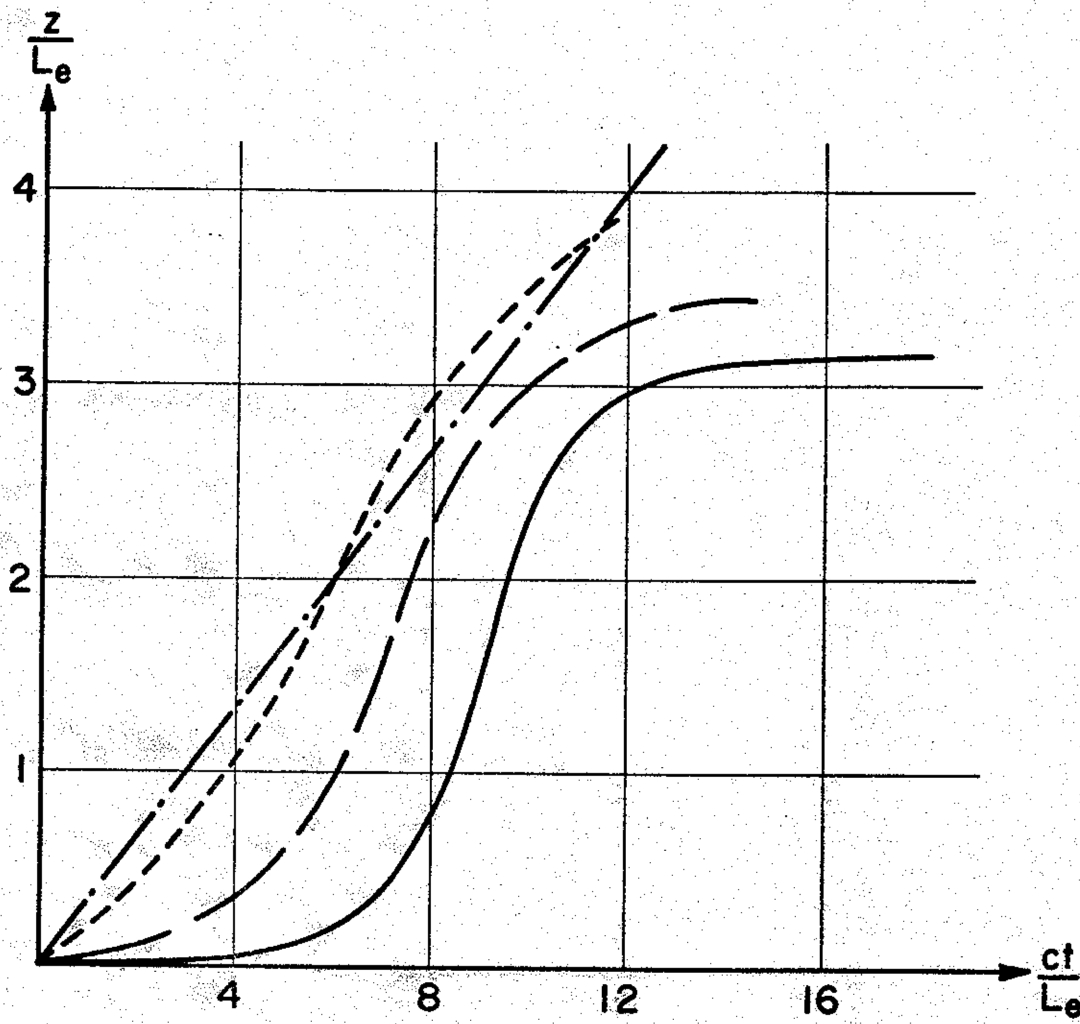


Figure 3c

$\frac{z}{L_e}$ Vs $\frac{ct}{L_e}$ for Electrons with $p_\theta^2 = 0$,
$50(mcL_e)^2$, $90(mcL_e)^2$, and $p_{\theta max}^2 \approx 101(mcL_e)^2$.

———————— (solid)	$p_\theta^2 = 0$ (Plotted from r = 0 to r = b and back to r = 0)
——— ——— (long dash)	$p_\theta^2 = 50(mcL_e)^2$ } (Plotted from r_1 to r_2 and back to r_1)
- - - - - - (short dash)	$p_\theta^2 = 90(mcL_e)^2$ } (Plotted from r_1 to r_2 and back to r_1)
——— · ——— (dash-dot)	$p_\theta^2 = p_{\theta max}^2$ $(z \approx \frac{ct}{3})$

In this equation, $K = \pi\epsilon_o\left(\frac{mc^2}{e}\right)^2$ = 7.28 Joules/meter, and we have used

$$\int_o^y xdxI_1{}^2(x) = yI_1(y)I_o(y) - \frac{y^2}{2}\,[I_o{}^2(y) - I_1{}^2(y)]. \tag{2.69}$$

Suppose the beam source is able to supply W Joules/meter (e.g. 2 kilojoules in a 10 meter long beam is 200 Joules/meter). Let α_w be defined by

$$W \equiv \alpha_w\gamma_o{}^2K\beta_z{}^2I_o{}^2\left(\frac{b}{L_e}\right) \equiv \alpha_w K(\gamma_o{}^2-1) \quad . \tag{2.70}$$

Part of W is in particle kinetic energy, U_p, and the rest is in the magnetic field. Let α_p and α_f be the field and particle contributions to α_w. Then

$$U_p = \int_o^b (\gamma_o-1)mc^2n_e(r)2\pi rdr = (\gamma_o-1)\gamma_o K\frac{b^2}{L_e{}^2} \equiv \alpha_p(\gamma_o{}^2-1)K. \tag{2.71}$$

The minimum magnetic field energy occurs when a = b, for which $\alpha_f \equiv \alpha_{min}$:

$$\alpha_{min} = \frac{b}{L_e}\left(\frac{I_1\left(\frac{b}{L_e}\right)}{I_o\left(\frac{b}{L_e}\right)} - \frac{b}{2L_e}\left(1-\frac{I_1{}^2\left(\frac{b}{L_e}\right)}{I_o{}^2\left(\frac{b}{L_e}\right)}\right)\right) \quad . \tag{2.72}$$

For the case considered above, γ_o = 2, I = 10^5 amperes ($\frac{b}{L_e}$ = 7.3), we obtain α_p = 35.5K and α_{min} = 3.13K, for a total minimum necessary energy of 845 Joules/meter. Note that this beam is a very efficient user of energy -- most of it is in particle energy if a = b. By contrast, for a uniform beam

$$\frac{U}{U_p} = \frac{I}{17000} \; \frac{(\gamma_o+1)^{\frac{1}{2}}}{4\gamma_o} \tag{2.73}$$

Therefore, if it could exist at 10^5 amperes and $\gamma_o = 2$, the uniform beam would have more energy tied up in fields than in particle motion. Suppose a > b, but $\frac{a-b}{b} << 1$. Then expanding the logarithm in Eq. (2.68), we obtain

$$\frac{a-b}{b} \simeq \alpha_f \left(\frac{L_e}{b}\right)^2 \frac{I_o{}^2\left(\frac{b}{L_e}\right)}{I_1{}^2\left(\frac{b}{L_e}\right)} - \frac{L_e}{b}\,\frac{I_o\left(\frac{b}{L_e}\right)}{I_1\left(\frac{b}{L_e}\right)} - \frac{b}{2L_e}\left[\frac{I_o{}^2\left(\frac{b}{L_e}\right)}{I_1{}^2\left(\frac{b}{L_e}\right)} - 1\right] . \tag{2.74}$$

The asymptotic expansions for I_o and I_1 in both (2.72) and (2.74) enable us to write

$$\frac{a-b}{b} \simeq \left(\frac{L_e}{b}\right)^2 \left(1 + \frac{L_e}{b} + \left(\frac{L_e}{b}\right)^2\right)(\alpha_f - \alpha_{min}). \tag{2.75}$$

Then for $\frac{b}{L_e} = 7.3$, and $\frac{a-b}{b} = 0.1$, $\alpha_f = 7.7$. Thus, a tube radius 10% greater than the beam radius results in more than double the field energy being required.

Up until this point, we have had no way to fix the beam radius given the beam current and electron energy; I/I_A merely fixes b/L_e. However, if the beam is in a drift tube of known radius, a, Eq. (2.75) enables us to fix b given the source energy. For example, if W = 1000 Joules/meter, $\gamma_o = 2$ and I = 10^5 amperes, then 155 Joules/meter are available for fields outside of $r = b$. This uniquely determines b at about .85a. It also implies that the beam would be hindered from pinching to a smaller radius than .85a by lack of sufficient energy. In fact, this tendency against pinching would be stronger, the higher the current, since Eq. (2.75) can also be written approximately as

$$\frac{a-b}{b} \simeq \frac{1}{4} \frac{I_A^{\,2}}{I^2} (\alpha_f - \alpha_{min}) \quad . \tag{2.76}$$

Although there are many beam models which can carry arbitrarily large currents (for example, Bennett's[1] and Benford's[19]), the one we have been considering is particularly interesting in that it is monoenergetic, and it is confined to a finite radius. Both of these are characteristic to some extent of most high current beam experiments to date. In addition, the current density is confined to a shell near the edge of the beam, and L. P. Bradley and J. C. Ingraham have observed high current beams which exhibit this characteristic.[20]

More generally we could superimpose beams such as we have considered with different values of P_z. An example is the electron distribution function

$$fe = \frac{c^2}{2\pi\varepsilon_e} \delta(H-\varepsilon_e)[n_1(0)\delta(P_z-\gamma_o mV_1) + n_2(0)\delta(P_z+\gamma_o mV_2)], \tag{2.77}$$

where $n_1(0) \ll n_2(0)$ and V_1 near c. The result would be a fast core carrying current below I_A, and a very slowly moving "halo" carrying most of the current, in which particles without angular momentum would be travelling backwards over part of their orbits, much like trajectory d in Fig. 1.

III. MAGNETIC NEUTRALIZATION

We now take up the notion of magnetic neutralization of an electron beam by a background plasma. We will develop a model in this chapter which indicates that cancellation of the beam current by large numbers of slowly counterstreaming electrons from a background plasma can be expected to occur. We assume the existence of a three dimensionally infinite, uniform, charge neutral, field free plasma consisting of mobile electrons and immobile ions. An electron beam is assumed to be moving through the plasma with velocity $\underline{v}_o$, the magnitude of which is large compared to the thermal velocity of the background plasma. At the initial time, $t = 0$, the beam extends from $z' = -\infty$ to $z' = 0$ (a primed coordinate indicating the laboratory frame of reference) along the z' axis, and it is neither electrostatically nor magnetically neutralized. We require that the beam's effect on the background plasma be small so that linear perturbation theory is valid, and then we consider the perturbed plasma motion in detail. Our results will, therefore, be valid for plasma density large compared to the beam density. The motion of the beam is assumed unaffected by the interaction. (We ignore the obvious problem of the two stream instability because the experiments which we are attempting to explain don't seem to be bothered by it.)

We solve this problem here with cold plasma two mass approximation relativistic fluid equations. This method enables us to extract the essential physics with a minimum of algebraic complication. In

Appendix B we show that using the relativistic Vlasov Equation with a two mass approximation Maxwellian gives the same result in the cold plasma limit. In the present treatment, we will see that the use of the two mass approximation involves dropping terms of order v_e^2/c^2, where v_e is the plasma electron thermal velocity, and c is the velocity of light. Therefore, retaining the pressure term in the momentum conservation equation would be inconsistent for beam velocities near c. Similarly, using the two mass approximation Maxwellian in the Vlasov treatment is certainly not inconsistent with the cold plasma limit.

We attack this problem in the rest frame of the beam, in which plasma is streaming by the beam with velocity $-v_o\hat{e}_z$. (An unprimed coordinate is a beam-at-rest frame coordinate.) In this frame, the beam stretches from $z = -\infty$ to $z = 0$ for all time and produces no magnetic field. We start from the Vlasov Equation with a phenomenological relaxation term,

$$\frac{\partial F(\underline{p})}{\partial t} + \underline{v}\cdot\frac{\partial F(\underline{p})}{\partial \underline{x}} - e(\underline{E}+\underline{v}\times\underline{B})\cdot\frac{\partial F(\underline{p})}{\partial \underline{p}} = -\frac{\left(F(\underline{p})-f_o(\underline{p})\right) - \frac{\delta n}{n} f_o(\underline{p})}{\tau} \quad . \qquad (3.1)$$

The relaxation term, much like that in the Krook-Bhatnager-Gross Equation,[21] is constructed to conserve particles locally, since δn, the perturbed plasma number density due to the beam-plasma interaction, is related to the "total" and unperturbed plasma electron momentum distribution functions $F(\underline{p})$ and $f_o(\underline{p})$, respectively, by

$$\delta n = n \int [F(\underline{p}) - f_o(\underline{p})]\, d\underline{p} \quad . \qquad (3.2)$$

The unperturbed plasma density is n, and τ is a phenomenological relaxation time. The first two moments of Eq. (3.1) are, assuming a cold plasma,

$$\frac{\partial N}{\partial t} + \nabla \cdot N\underline{V} = 0 \quad , \tag{3.3}$$

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla\right) \underline{P} = -e(\underline{E} + \underline{V}\times\underline{B}) - \frac{\underline{P} - \underline{p}_o}{\tau} \quad . \tag{3.4}$$

$N(\underline{x},t)$, $\underline{V}(\underline{x},t)$ and $\underline{P}(\underline{x},t)$ are the electron "fluid" density, velocity and momentum, respectively, and $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields. The electron charge is $-e$. At $t = 0$, the electron fluid quantities $N,\underline{V},\underline{P}$, have their unperturbed values n, $-v_o\hat{e}_z$, and $\underline{p}_o = -\gamma_o m v_o \hat{e}_z$, respectively. The electron rest mass is m and $\gamma_o = (1-v_o^2/c^2)^{-\frac{1}{2}}$. After $t = 0$, these quantities suffer perturbations due to the interaction with the beam so that

$$\begin{aligned} N &= n + \delta n \\ \underline{V} &= -v_o\hat{e}_z + \delta\underline{v} \\ \underline{P} &= -\gamma_o m v_o \hat{e}_z + \delta\underline{p} \quad . \end{aligned} \tag{3.5}$$

There are no applied fields, so $\underline{E}$ and $\underline{B}$ have only perturbation contributions $\delta\underline{E}$, and $\delta\underline{B}$, and only $\delta\underline{E}$ exists at $t = 0$ in the beam-at-rest frame. The linearized fluid equations for the perturbed quantities are therefore

$$\frac{\partial \delta n}{\partial t} + n\nabla\cdot\delta\underline{v} - v_o \frac{\partial}{\partial z} \delta n = 0 \tag{3.6}$$

$$\left(\frac{\partial}{\partial t} - v_o \frac{\partial}{\partial z}\right) \delta \underline{p} = -e(\delta \underline{E} - v_o \hat{e}_z \times \delta \underline{B}) - \frac{\delta \underline{p}}{\tau} \qquad (3.7)$$

We also have Maxwell's equations for the field quantities in terms of the plasma quantities:

$$\nabla \cdot \delta \underline{E} = \frac{\delta n e}{\varepsilon_o} \qquad (3.8)$$

$$\nabla \times \delta \underline{B} = \mu_o \, \delta \underline{j} + \frac{1}{c^2} \frac{\partial \delta \underline{E}}{\partial t} \qquad (3.9)$$

$$\nabla \times \delta \underline{E} = - \frac{\partial \delta \underline{B}}{\partial t} \qquad (3.10)$$

where $\delta \underline{j}$ is the background plasma current density due to the interaction. In order to close this set of equations, we still need a relationship between $\delta \underline{v}$ and $\delta \underline{p}$. The "total" quantities N, $\underline{V}$ and $\underline{P}$ and the perturbed quantities δn, $\delta \underline{v}$ and $\delta \underline{p}$ are defined in terms of $F(\underline{p})$ by

$$N = \int F(\underline{p}) d\underline{p} = n + \delta n \qquad (3.11a)$$

$$N\underline{V} = \int \underline{v} F(\underline{p}) d\underline{p}, \qquad N\delta \underline{v} = \int (\underline{v} + v_o \hat{e}_z) F(\underline{p}) d\underline{p} \qquad (3.11b)$$

$$N\underline{P} = \int \underline{p} F(\underline{p}) d\underline{p}, \qquad N\delta \underline{p} = \int (\underline{p} - \underline{p}_o) F(\underline{p}) d\underline{p} \quad , \qquad (3.11c)$$

where $\underline{v}$ and $\underline{p}$ are the velocity and momentum of an individual electron. Thus,

$$N\underline{V} = N\delta \underline{v} - N v_o \hat{e}_z \quad . \qquad (3.12)$$

But $\underline{p} = \gamma m\underline{v}$ for each electron, where $\gamma = (1+p^2/m^2c^2)^{-\frac{1}{2}}$.

A Taylor expansion of γ under the assumption $|\underline{p} - \underline{p}_o| << mc$ enables us to obtain

$$N\underline{V} = \int \frac{\underline{p}}{\gamma m} F(\underline{p}) d\underline{p} = \frac{N}{\gamma_o m}\left[\underline{p}_o + \delta\underline{p} - \frac{v_o^{\,2}\delta p_z \hat{e}_z}{c^2}\right] + 0\left[\int \frac{(\underline{p}-\underline{p}_o)^2}{m^2c^2} F(\underline{p}) d\underline{p}\right] . \quad (3.13)$$

Therefore

$$\delta\underline{v} \simeq \frac{\delta\underline{p}}{\gamma_o m} - \frac{v_o^{\,2}}{c^2}\frac{\delta p_z \hat{e}_z}{\gamma_o m} = \frac{\delta p_x \hat{e}_x + \delta p_y \hat{e}_y}{\gamma_o m} + \frac{\delta p_z \hat{e}_z}{\gamma_o^{\,3} m} \quad (3.14)$$

which is a statement of the two mass approximation. The terms we have dropped from Eq. (3.13) to get (3.14) are of the same order as the pressure term in the momentum equation would have been had we kept it.

The Fourier-Laplace transform, defined by the operator

$$\int_0^\infty dte^{-st} \int_{-\infty}^{\infty} d\underline{x}e^{-i\underline{k}\cdot\underline{x}} ,$$

of the linearized fluid equations and the last two of Maxwell's Equations are

$$(s-ik_z v_o)\delta n + i\underline{k}\cdot\delta\underline{v}n = 0 \quad (3.15)$$

$$(s-ik_z v_o + 1/\tau)\delta\underline{p} = -e(\delta\underline{E} - v_o \hat{e}_z \times \delta\underline{B}) \quad (3.16)$$

$$-s\delta\underline{B} = i\underline{k} \times \delta\underline{E} \quad (3.17)$$

$$i\underline{k} \times \delta\underline{B} = \mu_o \delta\underline{j} + \frac{s\delta\underline{E}}{c^2} - \frac{1}{c^2}\delta\underline{E}(\underline{k},\ t = 0) \quad . \quad (3.18)$$

The Laplace and Fourier transform variables are s and $\underline{k}$, respectively, where $k_z = \underline{k}\cdot\hat{e}_z$. δn, $\delta\underline{v}$, $\delta\underline{p}$, $\delta\underline{E}$ and $\delta\underline{B}$ are now all functions of $\underline{k}$ and s, and we have used $\delta n(t=0) = \delta\underline{p}(t=0) = \delta B(t=0) = 0$. The plasma response current $\delta\underline{j}$ is related to δn, δv and $\delta\underline{E}$ by

$$\delta\underline{j} = -e\delta(NV) = -e(n\delta\underline{v} - v_o\hat{e}_z\delta n) \equiv \underline{\underline{\sigma}}\cdot\delta\underline{E}, \qquad (3.19)$$

where $\underline{\underline{\sigma}}$ is the response "conductivity" tensor. Finally, we need $\delta\underline{E}(\underline{k}, t=0)$, which we obtain from the Fourier transform of Eq. (3.8) evaluated at t=0:

$$\delta\underline{E}(k,\ t=0) = -\frac{i\underline{k}\rho_b(\underline{k})}{\varepsilon_o k^2}\ . \qquad (3.20)$$

ρ_b, the charge density of the beam, is the only charge density at t=0. By our assumptions, it is not a function of time in the beam-at-rest frame.

Eqs. (3.14)-(3.20) constitute a closed set of equations, and in Appendix A, we solve them. The results for $\delta\underline{j}$ and $\delta\underline{E}$ in Fourier-Laplace transform space are

$$\delta j_1 = \frac{i\rho_b(\underline{k})\omega_p^{\ 2}}{kD_s}\left\{\left(1 - \frac{v_o^{\ 2}k_z^{\ 2}}{c^2k^2}\right)\left(k^2 + \frac{s^2}{c^2} + \frac{\omega_p^{\ 2}(s-ik_zv_o)}{c^2(s-ik_zv_o+1/\tau)}\right)\right.$$

$$\left. - \frac{\omega_p^{\ 2}k_\perp^{\ 2}v_o^{\ 2}(s-ik_zv_o)}{k^2c^4(s-ik_zv_o+1/\tau)}\right. \qquad (3.21a)$$

$$\delta j_2 = -\frac{\omega_p{}^2 k_\perp v_o \rho_b(\underline{k})}{ksD_s}\left(1+\frac{ik_z v_o s}{k^2c^2}\right)\left(k^2+\frac{s^2}{c^2}\right) \tag{3.21b}$$

$$\delta j_3 = 0 \tag{3.21c}$$

$$\delta E_1 = -\frac{i\rho_b(k)}{s\varepsilon_o kD_s}\left\{\left(k^2+\frac{s^2}{c^2}\right)(s+1/\tau-ik_z v_o)(ik_z v_o - s)\right.$$

$$\left. -\frac{\omega_p{}^2}{c^2}(s-ik_z v_o)^2\left[1-\frac{k_\perp{}^2 v_o{}^2\left(1+\frac{s^2}{k^2c^2}\right)}{(s-ik_z v_o)^2}\right]\right\} \tag{3.22a}$$

$$\delta E_2 = \frac{\omega_p{}^2 k_\perp v_o}{kc^2\varepsilon_o D_s}\left(1+\frac{ik_z v_o s}{k^2c^2}\right)\rho_b(\underline{k}) \tag{3.22b}$$

$$\delta E_3 = 0 \tag{3.22c}$$

where

$$D_s = \left\{(s+1/\tau-ik_z v_o)(ik_z v_o - s) - \omega_p{}^2\left(1-\frac{v_o{}^2}{c^2}\frac{k_z{}^2}{k^2}\right)\left(k^2+\frac{s^2}{c^2}\right)\right.$$

$$\left. -\frac{\omega_p{}^2}{c^2}(s-ik_z v_o)^2\left[1-\frac{k_\perp{}^2 v_o{}^2\left(1+\frac{s^2}{k^2c^2}\right)}{(s-ik_z v_o)^2}\right] - \frac{\omega_p{}^4(s-ik_z v_o)}{\gamma_o{}^2c^2(s+1/\tau-ik_z v_o)}\right\}. \tag{3.23}$$

The 1, 2 and 3 directions are defined by

$$\hat{e}_1 = \frac{\underline{k}}{k}, \qquad \hat{e}_3 = \frac{\underline{k}\times\hat{e}_z}{k_\perp}, \qquad \hat{e}_2 = \hat{e}_3\times\hat{e}_1 , \tag{3.24}$$

and $k_\perp$ is the component of $\underline{k}$ perpendicular to $\hat{e}_z$. $\delta\underline{B}$ is easily obtained from Eqs. (3.22) and (3.17), giving only $\delta B_3 \neq 0$. The perturbed charge density, $\delta\rho = -e\delta n$, is most easily obtained from Eq. (3.21a) through the Fourier-Laplace transform of the charge continuity equation,

$$s\delta\rho + ik\delta j_1 = 0 \quad . \tag{3.25}$$

We are interested in the behavior of our beam plasma system after the initial transients (and presumably the effects of our artificial initial conditions) have died down. To obtain this behavior, we take advantage of the Final-Value Theorem of Laplace transform theory, which enables us to write for any quantity, δQ,

$$\lim_{t\to\infty} \delta Q(t) = \lim_{s\to 0} s\delta Q(s). \tag{3.26}$$

(Note that we differentiate between a quantity and its transform only by exhibiting the appropriate variable as an argument.) There results

$$\lim_{t\to\infty} \delta j_1 = \lim_{t\to\infty} \delta E_2 = 0 \tag{3.27a}$$

$$\lim_{t\to\infty} \delta j_2(\underline{k}) = -\frac{\omega_p{}^2 k_\perp v_o \rho_b(\underline{k}) k}{D_o} \tag{3.27b}$$

$$\lim_{t\to\infty} \delta E_1(\underline{k}) = \frac{ikv_o{}^2\rho_b(\underline{k})}{\varepsilon_o D_o}\left[k_z{}^2\left(1+\frac{i}{k_z v_o \tau}\right) + \frac{\omega_p{}^2}{c^2}\right] \tag{3.27c}$$

$$\lim_{t\to\infty} \delta B_3(\underline{k}) = -\frac{i\rho_b(\underline{k})\omega_p{}^2 k_\perp v_o}{\varepsilon_o c^2 D_o} \tag{3.27d}$$

$$\lim_{t\to\infty} \delta\rho\ (\underline{k}) = \frac{\rho_b(\underline{k})\omega_p{}^2}{D_o}\left\{\left[k^2 - \frac{v_o{}^2}{c^2} k_z{}^2\right] + \frac{\omega_p{}^2}{\gamma_o{}^2 c^2(1+i/k_z v_o \tau)}\right\}, \tag{3.27e}$$

where

$$D_o \equiv D_s(s=0) = v_o{}^2 \left[k_z{}^2 k^2 + \frac{ik_z}{v_o\tau} \quad k^2 - \frac{\omega_p{}^2 k^2}{\gamma_o{}^2 v_o{}^2}\right.$$

$$\left. + \frac{\omega_p{}^2}{c^2} \quad k_z{}^2 - \frac{\omega_p{}^4}{\gamma_o{}^2 c^2 v_o{}^2(1+i/k_z v_o \tau)}\right] . \tag{3.28}$$

In order to obtain the spatial variation of a perturbed quantity, δQ, we must invert the Fourier Transform:

$$\delta Q(\underline{x},t\to\infty) = \int \frac{d^3k e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^3} \left[\lim_{t\to\infty} \delta Q(\underline{k})\right] . \tag{3.29}$$

For ρ_b, we choose a uniform beam of radius b and electron density n_b:

$$\rho_b(\underline{x}) = \begin{cases} -n_b e & z\leq 0,\ r\leq b \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

The Fourier inversions of the quantities in Eqs. (3.27) for this $\rho_b(\underline{x})$ are obtained in Appendix A. These results, in the beam-at-rest frame, valid for a weakly collisional plasma ($\omega_p\tau >> 1$) are, for $z < 0$:

$$\delta j_r = n_b e v_o \frac{\gamma_o \omega_p b}{v_o} \sin \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} I_1\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ I_1\left(\frac{\omega_p b}{v_o}\right) K_1\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \tag{3.31a}$$

$$\delta j_z = -n_b e v_o \left(\gamma_o^2 \frac{\omega_p b}{c} \left\{ \begin{matrix} I_o\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ \\ -I_1\left(\frac{\omega_p b}{c}\right) K_o\left(\frac{\omega_p r}{c}\right) \end{matrix} \right\} \right.$$

$$\left. - \cos \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \gamma_o^2 \frac{\omega_p b}{v_o} \left\{ \begin{matrix} I_o\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ \\ -I_1\left(\frac{\omega_p b}{v_o}\right) K_o\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \right), \tag{3.31b}$$

$$\delta B_\theta = -\mu_o n_b e v_o \left(\gamma_o^2 b \left\{ \begin{matrix} I_1\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ \\ I_1\left(\frac{\omega_p b}{c}\right) K_1\left(\frac{\omega_p r}{c}\right) \end{matrix} \right\} \right.$$

$$\left. -\gamma_o^2 b \cos \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} I_1\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ \\ -I_1\left(\frac{\omega_p b}{v_o}\right) K_1\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \right) \tag{3.31c}$$

$$\delta E_r = \frac{n_b eb}{\varepsilon_o} \left\{ \gamma_o^2 \frac{v_o^2}{c^2} \left\{ \begin{matrix} I_1\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ \\ I_1\left(\frac{\omega_p b}{c}\right) K_1\left(\frac{\omega_p r}{c}\right) \end{matrix} \right\} \right.$$

$$\left. -\gamma_o^2 \cos \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} I_1\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ I_1\left(\frac{\omega_p b}{v_o}\right) K_1\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \right\}, \qquad (3.31d)$$

$$\delta E_z = -\frac{n_b eb}{\varepsilon_o} \gamma_o \sin \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} \frac{v_o}{\omega_p b}\left[1 - \frac{\omega_p b}{v_o} I_o\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right)\right] \\ \\ I_1\left(\frac{\omega_p b}{v_o}\right) K_o\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \qquad (3.31e)$$

$$\delta\rho = n_b e \left\{ \left\{ \begin{matrix} 1 - \cos\frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \\ \\ 0 \end{matrix} \right\} + (\gamma_o^2 - 1) \left\{ \begin{matrix} \frac{\omega_p b}{c} I_o\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ \\ -\frac{\omega_p b}{c} I_1\left(\frac{\omega_p b}{c}\right) K_o\left(\frac{\omega_p r}{c}\right) \end{matrix} \right\} \right.$$

$$\left. -(\gamma_o^2 - 1) \frac{\omega_p b}{v_o} \cos \frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} I_o\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ \\ -I_1\left(\frac{\omega_p b}{v_o}\right) K_o\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \right\} . \qquad (3.31f)$$

The upper (lower) line is for $r<b$ ($r>b$), and the quantities, δQ, have arguments ($\underline{x}, t\to\infty$). All of these quantities are zero for $z>0$. I_m and K_m are modified Bessel functions of the first and second kind, respectively, and order m. In these results, a contribution to each perturbed quantity of order $e^{-\frac{\omega_p}{c}|z|}$, which is therefore significant only within c/ω_p of $z=0$, has been dropped. (Note that $\omega_p\tau>>1$ implies $c/\omega_p<<2v_o\tau$ for v_o near c.) This contribution is discussed in Appendix A. In addition, as mentioned in Appendix A, we have also dropped a collisional effect pointed out to us by R. E. Lee and R. N. Sudan[22] as it is unimportant in the parameter regime of particular interest to us.

Let us now look at a few of the characteristics of this solution in the beam-at-rest frame. Firstly, we note that several of the perturbed quantities are discontinuous across $r=b$. This is due to the discontinuous beam model and the cold plasma assumption. (Retaining the strongly damped terms - $0(e^{-\frac{\omega_p}{c}|z|})$ - would result in all quantities being continuous through $z = 0$.) We can easily calculate that the net axial current in the entire beam-plasma system is 0, by virtue of the integrals

$$\int rI_o(r)dr = rI_1(r), \qquad \int rK_o(r)dr = -rK_1(r). \tag{3.32}$$

For $\frac{\omega_p b}{c} >> 1$, the current density is confined to a sheath of thickness $\frac{c}{\omega_p}$ around $r = b$ since[15]

$$I_{0,1}\left(\frac{\omega_p r}{c}\right)K_{0,1}\left(\frac{\omega_p b}{c}\right) \sim \frac{c\exp[-\frac{\omega_p}{c}|b-r|]}{2\omega_p(rb)^{\frac{1}{2}}} \tag{3.33}$$

The same thing can be said about the rest of the quantities except for $\delta\rho$ and δE_z. We are, therefore, led to the following physical interpretation: The electron "fluid" flowing in toward the beam from the right does not know the beam is there until it reaches $z = 0$ (actually $z \sim \frac{c}{\omega_p}$, had we not dropped the strongly damped term). Suddenly encountering the beam, the electron fluid expands within the beam (that is - the density decreases as plasma electrons are thrown out of the beam) in an attempt to neutralize the bulk of the beam charge density. A standing wave is set up as a result of this; this wave is simply a damped plasma oscillation in the laboratory frame. When the electron fluid oscillation has been damped ($|z|>2v_o\tau$), the bulk of the beam charge density has been neutralized by a net ion density of n_b having been left behind (the first term for $r<b$ in Eq. (3.31f). The excess electrons have been carried off to infinity since the $2\pi\int_0^\infty(\delta\rho+\rho_b)rdr$ is zero for $|z|>2v_o\tau$. The ions that have been left behind for $|z| > 2v_o\tau$ must be contributing a current density of $-n_b ev_o$ to δj_z, the magnitude of which is certainly small compared to this everywhere but within c/ω_p of $r=b$. The cancelling current is a result of a net acceleration in the $-z$ direction of the electron fluid within the beam. FromAppendix A, Eq. (A.40),

$$\delta v_z = -\frac{n_b}{n} v_o \left(\left\{ \begin{matrix} 1 - \cos\frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \\ \\ 0 \end{matrix} \right\} - \frac{\omega_p b}{c} \left\{ \begin{matrix} I_o\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ \\ -I_1\left(\frac{\omega_p b}{c}\right) K_o\left(\frac{\omega_p r}{c}\right) \end{matrix} \right\} \right.$$

$$\left. + \cos\frac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o\tau}} \left\{ \begin{matrix} \frac{\omega_p b}{v_o} I_o\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) \\ \\ -\frac{\omega_p b}{v_o} I_1\left(\frac{\omega_p b}{v_o}\right) K_o\left(\frac{\omega_p r}{v_o}\right) \end{matrix} \right\} \right), \quad z < 0. \tag{3.34}$$

As before, the upper (lower) line is $r<b$ ($r>b$), and $\delta v_z = 0$ for $z>0$. (In this, we have dropped the same contributions as in Eqs. (3.31).) It is clear that the electron current due to the first pair of braces in δv_z for $r<b$ is exactly that required to cancel the same term in $\delta\rho$, leaving only the sheath current density in δj_z. From Eq. (A.38) it is clear that δE_z is responsible for the acceleration of the electron fluid. Finally, we can see that unless $n_b \ll n$, δv_z and $\delta\rho$ will not be small compared to the unperturbed quantities in the electron plasma. Since we have used linear perturbation theory to obtain our solution, we must have $|\delta v_z| \ll v_o$ and $|\delta\rho| \ll ne$. Therefore, our solution can be valid only for $n_b \ll n$.

We now transform the complete solution into the laboratory frame of reference using the Lorentz Transformation. With primes denoting laboratory frame quantities, the necessary transformations are

$$z = \gamma_o(z' - v_o t'), \quad r = r' \tag{3.35a}$$

$$n = \gamma_o n', \quad n_b' = \gamma_o n_b \tag{3.35b}$$

$$\omega_p^{\,2} \equiv \frac{ne^2}{\gamma_o m\varepsilon_o} = \frac{n'e^2}{m\varepsilon_o} \equiv \omega_p'^{\,2} \tag{3.35c}$$

$$\delta j_z' = \gamma_o(\delta j_z + v_o\delta\rho), \quad \delta j_r' = \delta j_r \tag{3.35d}$$

$$\delta\rho' = \gamma_o\left(\delta\rho + \frac{v_o\delta j_z}{c^2}\right) \tag{3.35e}$$

$$\delta B_\theta' = \gamma_o\left(\delta B_\theta + \frac{v_o\delta E_r}{c^2}\right) \tag{3.35f}$$

$$\delta E_r' = \gamma_o(\delta E_r + v_o\delta B_\theta), \quad \delta E_z' = \delta E_z \quad . \tag{3.35g}$$

For $z'-v_ot'<0$, and large t', there results

$$\delta j_r' \simeq n_b'ev_o \sin\frac{\omega_p'(z'-v_ot')}{v_o}\, e^{\frac{z'-v_ot'}{2v_o\tau'}} \begin{Bmatrix} \dfrac{\omega_p'b}{v_o} I_1\left(\dfrac{\omega_p'r'}{v_o}\right)K_1\left(\dfrac{\omega_p'b}{v_o}\right) \\ \dfrac{\omega_p'b}{v_o} I_1\left(\dfrac{\omega_p'b}{v_o}\right)K_1\left(\dfrac{\omega_p'r'}{v_o}\right) \end{Bmatrix} \tag{3.36a}$$

$$\delta j_z' + \begin{Bmatrix} -n_b'ev_o \\ \\ 0 \end{Bmatrix} \simeq -n_b'ev_o\left(\begin{Bmatrix} \dfrac{\omega_p'b}{c} I_o\left(\dfrac{\omega_p'r'}{c}\right) K_1\left(\dfrac{\omega_p'b}{c}\right) \\ -\dfrac{\omega_p'b}{c} I_1\left(\dfrac{\omega_p'b}{c}\right) K_o\left(\dfrac{\omega_p'r'}{c}\right) \end{Bmatrix} \right.$$

$$\left. + \cos\frac{\omega_p'(z'-v_ot')}{v_o}\quad e^{\frac{z'-v_ot'}{2v_o\tau}} \begin{Bmatrix} \left[1-\dfrac{\omega_p'b}{v_o} I_o\left(\dfrac{\omega_p'r'}{v_o}\right)K_1\left(\dfrac{\omega_p'b}{v_o}\right)\right] \\ \\ \dfrac{\omega_p'b}{v_o} I_1\left(\dfrac{\omega_p'b}{v_o}\right) K_o\left(\dfrac{\omega_p'r'}{v_o}\right) \end{Bmatrix} \right)$$

(3.36b)

$$\delta B_\theta' \simeq -\mu_o n_b' e v_o b \begin{Bmatrix} I_1\left(\frac{\omega_p' r'}{c}\right) K_1\left(\frac{\omega_p' b}{c}\right) \\ \\ I_1\left(\frac{\omega_p' b}{c}\right) K_1\left(\frac{\omega_p' r'}{c}\right) \end{Bmatrix} \tag{3.36c}$$

$$\delta E_r' \simeq -\frac{n_b' e b}{\varepsilon_o} \cos \frac{\omega_p'(z'-v_o t')}{v_o} e^{\frac{z'-v_o t'}{2v_o\tau'}} \begin{Bmatrix} I_1\left(\frac{\omega_p' r'}{v_o}\right) K_1\left(\frac{\omega_p' b}{v_o}\right) \\ \\ I_1\left(\frac{\omega_p' b}{v_o}\right) K_1\left(\frac{\omega_p' r'}{v_o}\right) \end{Bmatrix} \tag{3.36d}$$

$$\delta E_z' \simeq -\frac{n_b' e b}{\varepsilon_o} \sin \frac{\omega_p'(z'-v_o t')}{v_o} e^{\frac{z'-v_o t'}{2v_o\tau'}} \begin{Bmatrix} \frac{v_o}{\omega_p' b}\left[1-\frac{\omega_p' b}{v_o} I_o\left(\frac{\omega_p' r'}{v_o}\right) K_1\left(\frac{\omega_p' b}{v_o}\right)\right] \\ \\ I_1\left(\frac{\omega_p' b}{v_o}\right) K_o\left(\frac{\omega_p' r'}{v_o}\right) \end{Bmatrix} \tag{3.36e}$$

$$\delta\rho' + \begin{Bmatrix} -n_b' e \\ \\ 0 \end{Bmatrix} \simeq -n_b' e \begin{Bmatrix} \cos \frac{\omega_p'(z'-v_o t')}{v_o} e^{\frac{z'-v_o t'}{2v_o\tau'}} \\ \\ 0 \end{Bmatrix} . \tag{3.36f}$$

The upper (lower) line is again for $r<b$ $(r>b)$. Note that to $\delta\rho'$ and $\delta j_z'$ above, we have added the beam charge and current densities; therefore, the exhibited quantities are the net charge and current densities. The contributions from the strongly damped terms have been dropped, and therefore, to this approximation, all of the above quantities are zero for $z'-v_o t'>0$.

In this frame it is clear that the net charge is zero away from the front of the beam, and that at a fixed $z'<v_o t'$, we have a simple damped plasma oscillation. As in the beam frame, for $\omega_p' b/c>>1$, the net current density is confined to a sheath of thickness c/ω_p'. Therefore, if a beam electron has left this sheath before it has gained much perpendicular energy -- $|v_z|>>|v_\perp|$ -- then we are justified in having said that the beam is unaffected by the interaction. We would, therefore, want the Larmor radius, R_L, of a beam electron in the maximum magnetic field to be large compared to c/ω_p'. This is nearly equivalent to the original $n_b<<n$ requirement:

$$R_L = \frac{\gamma_o m v_o}{e\delta B_\theta(r=b)} \cong \frac{\gamma_o m v_o}{e\mu_o n_b' e v_o b}\frac{2\omega_p' b}{c} >> \frac{c}{\omega_p'} \qquad (3.37)$$

or

$$n_b' << \frac{\omega_p'^2 2m\gamma_o}{\mu_o c^2 e^2} = 2\gamma_o n' \quad . \qquad (3.38)$$

Approximately the same inequality results from consideration of time scales. For the plasma to be able to charge and current neutralize the system before the beam has expanded significantly, we must have $\omega_p'^{-1} << \omega_{pb}'^{-1}$, ω'_{pb} being the beam electron plasma frequency. Except for the factor 2, relation (3.38) results. Thus, an arbitrarily large total beam current can be propagated in this model as long as the beam radius is sufficiently large that $n_b'<<n'$. As an indication of how the space dependence of the net current density changes with $\omega_p'b/c$, in Fig. 4, we plot the ratio of $\delta j_z'$ to $n_b'ev_o$ for $\omega_p'b/c = 1$ and 10.

If $b<<c/\omega_p'$, the total current within $r'=b$ for $|z'|>2v_o\tau'$ is[15]

$$-2\pi b^2 n_b' e v_o I_1\left(\frac{\omega_p' b}{c}\right) K_1\left(\frac{\omega_p' b}{c}\right) \simeq -n_b' e v_o \pi b^2 \ , \qquad (3.39)$$

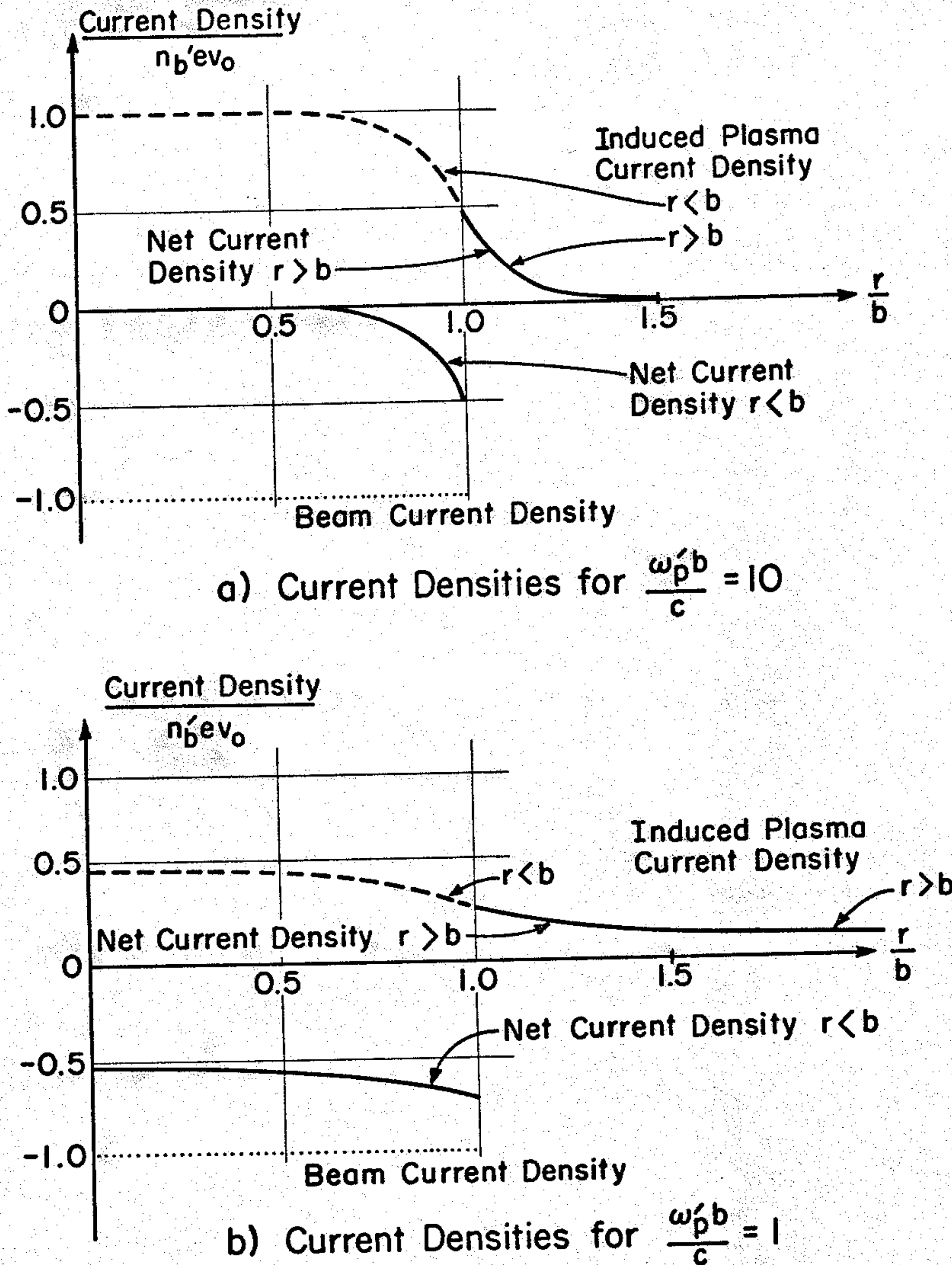


Figure 4 - Current Densities Relative to $n_b' e v_o$ for $z' >> 2 v_o \tau'$

which is the full beam current. The same magnitude net current with the opposite sign is flowing outside $r'=b$. Hence, in this limit, all of the return current being supplied by the plasma is outside the beam, and no magnetic neutralization occurs.

For the beam of Andrews, et al., a current of 10^5 amperes at $\gamma_o = 2$ in a radius of 5 cm gives $n_b' \approx 2\times10^{11}/cm^3$. An ambient pressure of .5 torr implies about $2\times10^{16}/cm^3$ neutral density. The plasma density at a point after a length L of beam has passed is approximately given by

$$n' \simeq n_b' \, n_o \, \sigma_I L \tag{3.40}$$

where n_o is the neutral density and σ_I is the appropriate effective ionization cross section. From $\sigma_I \simeq 2\times10^{-18}$ cm^2,[23]

$$n' \simeq 4 \, n_b' \, L \tag{3.41}$$

(L in meters). A pulse duration of 50 nanoseconds implies a beam length of order 15 meters. Therefore, for purposes of our theory, we assume $n' \gtrsim 2\times10^{12}$ ($>>n_b'$), so that $\omega_p' > 8\times10^{10}$/sec, $\frac{c}{\omega_p'} < .4$ cm and $\frac{b\omega_p'}{c} > 12$. Estimating τ' from the formula given by Rose and Clark,[24] we obtain about $1.5 \; 10^{-9}$ sec., so that $\omega_p'\tau' >> 1$ and $2v_o\tau' \sim 1$ meter. From these numbers, we see that the case of interest is $\frac{b\omega_p'}{c} >> 1$, in which a great deal of current neutralization is to be expected. For this case the length characteristic of the collisional damping effect found by Lee and Sudan[22] is at least of order $\frac{b\omega_p'}{c} 2v_o\tau' >> 2v_o\tau'$. Therefore, collisional damping should not dominate the current neutralization over most of the beam length in the high current beam propagation experiments to date.

IV. LONGITUDINAL GUIDE FIELD

In this section we take up the problem of a uniform electron beam of radius, b, infinite in the axial (z) direction, in the presence of a uniform, axial magnetic induction $\underline{B}_o = B_o\hat{e}_z$. We will find that if B_o is much larger than the beam's self magnetic induction, the electron beam can be expected to propagate.

For this problem, Cartesian coordinates (x, y, z) prove to be the most convenient. Therefore, we express the beam's self fields, Eq. (2.61f) and (2.61g), as

$$E_x = -\frac{Ne(1-f)x}{2\pi\varepsilon_o b^2} \quad , \quad E_y = -\frac{Ne(1-f)y}{2\pi\varepsilon_o b^2} \tag{4.1}$$

$$B_x = \frac{NeVy}{2\pi\varepsilon_o c^2 b^2} \quad , \quad B_y = -\frac{NeVx}{2\pi\varepsilon_o c^2 b^2} \quad . \tag{4.2}$$

V is the beam propagation velocity and N is again the number of electrons per meter of beam. We intend to use these fields, together with $\underline{B}_o$, in

$$\frac{d}{dt}\gamma m\underline{v} = -e(\underline{E}+\underline{v}\times\underline{B}) \tag{4.3}$$

under the assumptions which make a uniform beam self consistent:

$$|v_\perp| << V \tag{4.4a}$$

$$|v_z - V| << V \tag{4.4b}$$

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} \simeq \text{constant.} \tag{4.4c}$$

$\beta = \frac{V}{c}$ is assumed near one.

Under the above assumptions the equations of motion for a beam electron reduce approximately to

$$\gamma m\ddot{x} \simeq \frac{Ne^2[(1-f)-\beta^2]x}{2\pi\varepsilon_o b^2} - e\dot{y}B_o \tag{4.5a}$$

$$\gamma m\ddot{y} \simeq \frac{Ne^2[(1-f)-\beta^2]y}{2\pi\varepsilon_o b^2} + e\dot{x}B_o \tag{4.5b}$$

$$\gamma^3 m\ddot{z} \simeq 0. \tag{4.5c}$$

Dropped terms are of order $v_\perp/c$ or $|v_z - V|/c$. A dot represents a time derivative in these equations. Defining

$$\Omega_o = \frac{eB_o}{\gamma m} \tag{4.6a}$$

$$\Omega^2 = \frac{Ne^2}{\gamma 2\pi\varepsilon_o mb^2}(1-f-\beta^2) \equiv \frac{2\nu c^2}{\gamma b^2}(1-f-\beta^2) \tag{4.6b}$$

$$\zeta = x + iy, \tag{4.6c}$$

and multiplying Eq. (4.5b) by i and adding it to Eq. (4.5a), results in

$$\ddot{\zeta} = -\Omega^2\zeta + i\Omega_o\dot{\zeta} \; . \tag{4.7}$$

Looking for solutions of the form $Ae^{i\omega t}$, we obtain

$$\omega^2 - \Omega_o\omega - \Omega^2 = 0. \tag{4.8}$$

The two roots of this equation are

$$\omega_{\pm} = \frac{\Omega_o}{2} \left[1 \pm \left(1 + \left(\frac{2\Omega}{\Omega_o}\right)^2\right)^{\frac{1}{2}}\right] \tag{4.9}$$

If we take initial conditions $\zeta(t=0) = \zeta_o$ and $\dot{\zeta}(t=0) = 0$, then

$$\zeta(t) = \zeta_o \left(\frac{\omega_-}{\omega_- - \omega_+} e^{i\omega_+ t} + \frac{\omega_+}{\omega_+ - \omega_-} e^{i\omega_- t}\right) . \tag{4.10}$$

We can find x(t) and y(t) from the real and imaginary parts of $\zeta(t)$.

Consider the limit $\Omega_o{}^2 >> 4|\Omega^2|$. This is equivalent to

$$B_o >> 2\left(\frac{2\gamma}{\nu}\right)^{\frac{1}{2}} |B_{max}| \left(\frac{1-f-\beta^2}{\beta^2}\right) . \tag{4.11}$$

($|B_{max}|$ is the maximum value of $|B_x|$.) In this case,

$$\omega_+ \simeq \Omega_o [1 + (\frac{\Omega}{\Omega_o})^2] \tag{4.12a}$$

$$\omega_- \simeq - \frac{\Omega^2}{\Omega_o} \tag{4.12b}$$

and we obtain

$$\zeta(t) \simeq \zeta_o \left[(\frac{\Omega}{\Omega_o})^2 e^{i\Omega_o t} + (1 - \frac{\Omega^2}{\Omega_o{}^2}) e^{-i\frac{\Omega^2}{\Omega_o}t} \right] . \tag{4.13}$$

Hence, in the large axial field limit we have the sum of two rotations -- a high frequency gyration with radius $|\zeta_o|(\frac{\Omega}{\Omega_o})^2 << |\zeta_o|$ about a guiding center which slowly rotates around the beam axis with radius nearly $|\zeta_o|$. These are, of course, superimposed on the "uniform" motion in the z direction with velocity V.

The assumption that $|v_\perp| << V$, taking the worst case for $|\dot{\zeta}|$, yields the requirement

$$B_o >> \frac{|B_{max}|}{\beta^2} \quad (1-\beta^2-f) \quad . \tag{4.14}$$

This is much stronger than the inequality (4.11) for a high current beam. For f=1, the relation is $B_o >> |B_{max}|$ and we have a practical limit for neutralized beam current in this model since arbitrarily large guide fields are expensive. However, in principle, by applying a large enough guide field, arbitrarily large currents could be propagated without the occurrence of catastrophic pinching due to the beam's self field. If in fact, $|B_o| \approx |B_{max}|$, one would expect that flux would be conserved as beam pinching begins to occur. The axial field within the beam would then increase with the square of the original to final radius ratio, whereas the maximum self field increases only linearly with this ratio. We would then have the required inequality after a moderate amount of pinching. Therefore, the effect of a large axial guide field should be apparent even if the restriction (4.14) is not strongly satisfied.

In this discussion, we have assumed that the axial magnetic induction is uniform. However, the analysis should be applicable so long as the change in the guide field over a gyro-radius is small.

It also should be noticed that the perpendicular motion equation, (4.13), contains only Ω^2. This means that regardless of the sign of $1-f-\beta^2$, Eq. (4.13) holds; only the direction of the slow rotation changes. One can therefore apply it to a totally unneutralized beam (f=0) as well as to a neutralized beam (f=1). Applicability extends to magnetically neutralized beams as well with the substitution of $1-f-\beta^2(1-f_m)$ for $1-f-\beta^2$ in Ω^2. In this case, of course, less guide field is required since $|B_{max}|$ decreases.

V. DISCUSSION

In the three preceding sections, we presented three models of relativistic electron beams which allow the propagation of arbitrarily large currents within a finite radius. The three models avoid the catastrophic self-magnetic pinch exhibited by electrostatically neutralized high current uniform beams by three distinct physical mechanisms. The fully relativistic self-consistent equilibrium does it by concentrating the current density near the edge of the beam so that beam electrons have left high field regions before they have a chance to turn around on themselves. The initial value problem solution suggests that a beam propagating into a high density background plasma will avoid the self pinch problem by inducing plasma currents which cancel out the beam's self-magnetic field. Finally, adding the strong axial guide field to the uniform beam solves the problem by limiting radial excursions by a beam electron to small ones in the form of a rotation about the electron's guiding center, the radial position of which is approximately constant.

All three of our models serve to explain some experimental observations made to date on propagating high current relativistic electron beams. Experiments with an axial guide field being done by Bzura, Andrews and Fleischmann[25] with the guide field related to the maximum beam self field by $|B_o|>|B_{max}|$, at various ambient pressures, indicate that the guide field does help with beam propagation. Relatively slow beam propagation velocities observed both by Andrews, _et al._,[13] and by Yonas and Spence,[12] and the beams with a shell current density observed occasionally by Bradley and Ingram, seem to point to the non-uniform equilibrium. Finally, magnetic field measurements made on high $\frac{\nu}{\gamma}$ beams injected into drift regions at pressures above .1 torr indicate that partial magnetic neutralization takes place,[12,13] as predicted by the model of Section III. However, none of our models is adequate to explain all phenomena observed even in a single experiment. Except perhaps those with magnetic guide fields, experiments to date have not been performed in such a way that we should expect complete explanation by one of our physical principles. For example, no attempt has been made to start a high current beam off with a shell current density into a background plasma very nearly equal to the beam density. Nor has a systematic attempt been made to study beams propagating into high density, quiescent plasmas. Instead, experimental groups usually inject beams into neutral gas, and they have found that for significant beam propagation, ambient pressures of above .1 torr are necessary. As previously mentioned (Eq. 3.41), this means that the background plasma density, n', is continually building up during an experiment according to

$$n' \simeq 4n_b' L \qquad (6.1)$$

(at an ambient pressure of about .5 torr of air and with L in meters), where n_b' is the beam density. Thus, we can expect f = 1 after a half meter of beam has passed and $n' \simeq 10n_b'$ after about five meters. With such a rapid build-up of plasma in these experiments the model of Section II can at best be a state through which the beam-plasma-neutral gas system passes early in the interaction, on its way to becoming at least partially magnetically neutralized, as observed experimentally. This would also account for the relatively slow propagation velocities observed for the beam front.[12,13] Possibly the lingering effects of having the current density concentrated in a shell early in the pulse is the explanation of the Bradley and Ingram observations.[20]

APPENDIX A

We wish to take up here the solution of Eqs. (3.14) - (3.20) of Chapter II, and the inverse Fourier transformation of the elements of that solution. In order to facilitate this solution, we introduce the Cartesian coordinate system defined by

$$\hat{e}_1 = \frac{\underline{k}}{k} \;, \quad \hat{e}_3 = \frac{\underline{k} \times \hat{e}_z}{k_\perp} \;, \quad \hat{e}_2 = \hat{e}_3 \times \hat{e}_1 \;, \tag{A.1}$$

where $k_\perp$ is the component of $\underline{k}$ perpendicular to $\hat{e}_z$, and k is the magnitude of $\underline{k}$. We therefore have

$$\hat{e}_z \cdot \hat{e}_1 = \frac{k_z}{k} \;, \quad \hat{e}_z \cdot \hat{e}_2 = \frac{k_\perp}{k} \;, \quad \hat{e}_z \cdot \hat{e}_3 = 0 \;. \tag{A.2}$$

In this coordinate system, Eq. (3.15) becomes

$$\delta n = -\frac{nik\delta v_1}{s - ik_z v_o} \quad . \tag{A.3}$$

Eqs. (3.14), (3.19) and (A.3) together yield the components of $\delta\underline{j}$ in terms of the components of $\delta\underline{p}$:

$$\delta j_1 = -\frac{en\left\{\delta p_1\left(1 - \frac{v_o^{\,2} k_z^{\,2}}{k^2 c^2}\right)s - \frac{v_o^{\,2} k\, k_z s \delta p_2}{k^2 c^2}\right\}}{\gamma_o m (s - ik_z v_o)} \tag{A.4a}$$

$$\delta j_2 = -\frac{en}{\gamma_o m}\left\{\delta p_2\left[1 - \frac{v_o^{\,2} k^{\,2} s}{c^2 k^2 (s - ik_z v_o)}\right] + \delta p_1\left[\frac{ik\, v_o - s\frac{v_o^{\,2}}{c^2}\frac{k_z k_\perp}{k^2}}{s - ik_z v_o}\right]\right\} \tag{A.4b}$$

$$\delta j_3 = -\frac{en\delta p_3}{\gamma_o m} \quad . \tag{A.4c}$$

The components of $\delta \underline{p}$ are obtained in terms of those of $\delta \underline{E}$ by combining Eqs. (3.16) and (3.17) and expanding the cross product. In terms of the coordinate system defined above, there results

$$\delta \underline{p} = -\frac{e}{s}\frac{[(\delta E_1 s + ik_z v_o \delta E_2)\hat{e}_1 + (\delta E_2 \hat{e}_2 + \delta E_3 \hat{e}_3)(s - ik_z v_o)]}{(s - ik_z v_o + 1/\tau)} \quad . \tag{A.5}$$

The substitution of the components of $\delta \underline{p}$ into Eqs. (A.4) and the use of $\delta \underline{j} = \underline{\underline{\sigma}} \cdot \delta \underline{E}$ enable us to obtain the components of $\underline{\underline{\sigma}}$:

$$\sigma_{11} = \frac{\varepsilon_o \omega_p^{\,2} s \left(1 - \frac{v_o^{\,2}}{c^2}\frac{k_z^{\,2}}{k^2}\right)}{(s + 1/\tau - ik_z v_o)(s - ik_z v_o)} \tag{A.6a}$$

$$\sigma_{12} = \sigma_{21} = \frac{\varepsilon_o \omega_p^{\,2} ik_z v_o \left[1 + \frac{ik_z v_o s}{k^2 c^2}\right]}{(s - ik_z v_o + 1/\tau)(s - ik_z v_o)} \tag{A.6b}$$

$$\sigma_{22} = \frac{\varepsilon_o \omega_p^{\,2}(s - ik_z v_o)}{s(s - ik_z v_o + 1/\tau)}\left(1 - \frac{k_z^{\,2} v_o^{\,2}\left[1 + \frac{s^2}{k^2 c^2}\right]}{(s - ik_z v_o)^2}\right) \tag{A.6c}$$

$$\sigma_{33} = \frac{\varepsilon_o \omega_p^{\,2}(s - ik_z v_o)}{s(s - ik_z v_o + 1/\tau)} \tag{A.6d}$$

$$\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0. \tag{A.6e}$$

The electron plasma frequency is

$$\omega_p^{\,2} \equiv \frac{ne^2}{\gamma_o m \varepsilon_o} \quad . \tag{A.7}$$

The wave equation with sources for $\delta\underline{E}$ is obtained by combining Eqs. (3.17), (3.18) and (3.20) and expanding $\underline{k} \times (\underline{k} \times \delta\underline{E})$:

$$\left(k^2 + \frac{s^2}{c^2}\right)\delta\underline{E} - \underline{k}(\underline{k}\cdot\delta\underline{E}) + \frac{s}{\varepsilon_o c^2}\,\delta\underline{j} = -\frac{s\rho_b(\underline{k})i\underline{k}}{\varepsilon_o c^2 k^2} \quad . \tag{A.8}$$

Eliminating $\delta\underline{j}$ from this equation through $\delta\underline{j} = \underline{\underline{\sigma}}\cdot\delta\underline{E}$ results in

$$\underline{\underline{Y}} \cdot \delta\underline{E} = S_1\hat{e}_1, \tag{A.9}$$

where S, and $\underline{\underline{Y}}$ are defined by

$$S_1 \equiv -\frac{is\rho_b(\underline{k})}{\varepsilon_o c^2 k} , \tag{A.10}$$

$$\underline{\underline{Y}} \equiv \left(k^2 + \frac{s^2}{c^2}\right) \underline{\underline{1}} - k^2\hat{e}_1\hat{e}_1 + \frac{s}{\varepsilon_o c^2}\,\underline{\underline{\sigma}} \quad . \tag{A.11}$$

$\underline{\underline{1}}$ is the unit tensor. Considering Eqs. (A.6), (A.9) is equivalent to

$$\begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{pmatrix} \begin{pmatrix} \delta E_1 \\ \delta E_2 \\ \delta E_3 \end{pmatrix} = \begin{pmatrix} S_1 \\ 0 \\ 0 \end{pmatrix} . \tag{A.12}$$

This is readily solved for the components of $\delta\underline{E}$:

$$\delta E_1 = \frac{Y_{22}S_1}{Y_{11}Y_{22}-Y_{12}^2} \tag{A.13a}$$

$$\delta E_2 = -\frac{Y_{12}S_1}{Y_{11}Y_{22}-Y_{12}^2} \tag{A.13b}$$

$$\delta E_3 = 0. \tag{A.13c}$$

$\delta \underline{j} = \underline{\underline{\sigma}} \cdot \delta \underline{E}$ then gives for the components of $\delta \underline{j}$

$$\delta j_1 = \frac{(\sigma_{11}Y_{22}-\sigma_{12}Y_{12})S_1}{Y_{11}Y_{22}-Y_{12}^2} \tag{A.14a}$$

$$\delta j_2 = \frac{(\sigma_{12}Y_{22}-\sigma_{22}Y_{12})S_1}{Y_{11}Y_{22}-Y_{12}^2} \tag{A.14b}$$

$$\delta j_3 = 0 \quad . \tag{A.14c}$$

These quantities are given explicitly in Section III, Eqs. (3.21)-(3.23).

The time asymptotic behavior of these quantities is obtained from the Final-Value Theorem of Laplace transform calculus, quoted in Eq. (3.24). The results of this are given in the text in Eqs. (3.27). From Fig. 5 and Eqs. (A.1),

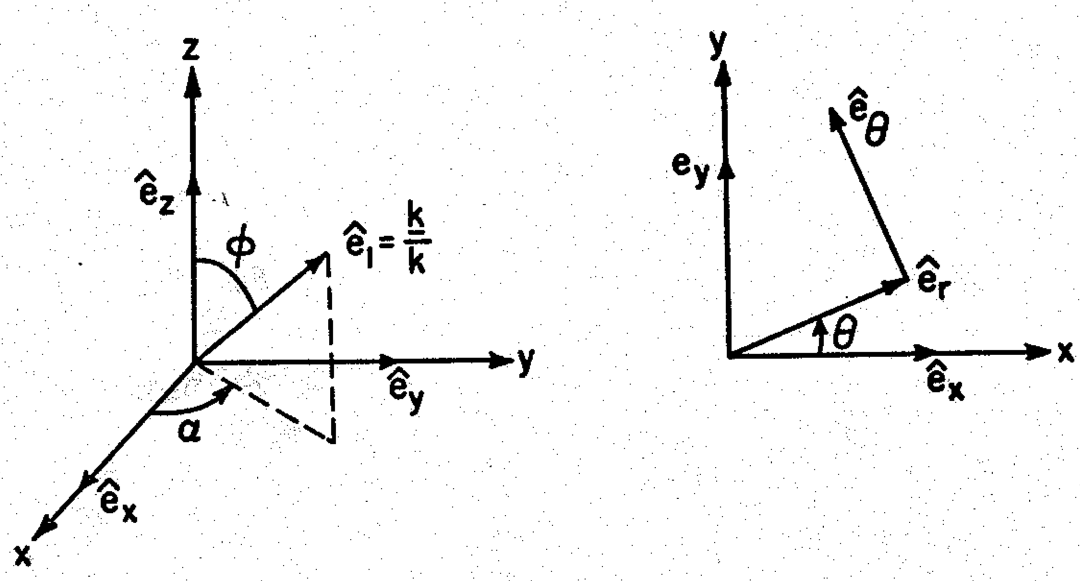
$$\hat{e}_1 = \hat{e}_r \frac{k}{k} \cos(\alpha-\theta) + \hat{e}_\theta \frac{k}{k} \sin(\alpha-\theta) + \hat{e}_z \frac{k_z}{k} , \tag{A.15a}$$

$$\hat{e}_2 = -\hat{e}_r \frac{k_z}{k} \cos(\alpha-\theta) - \hat{e}_\theta \frac{k_z}{k} \sin(\alpha-\theta) + \hat{e}_z \frac{k}{k} , \tag{A.15b}$$

$$\hat{e}_3 = \hat{e}_r \sin(\alpha-\theta) - \hat{e}_\theta \cos(\alpha-\theta). \tag{A.15c}$$

Therefore, the r, θ and z components of $\delta \underline{j}$, $\delta \underline{E}$, and $\delta \underline{B}$, and the charge density, $\delta\rho$, from Eqs. (3.27), are

$$\delta j_r(\underline{k}, t\to\infty) = \frac{\cos(\alpha-\theta) k_z k_\perp \omega_p^{\,2} v_o \rho_b(k)}{D_o} \tag{A.16a}$$



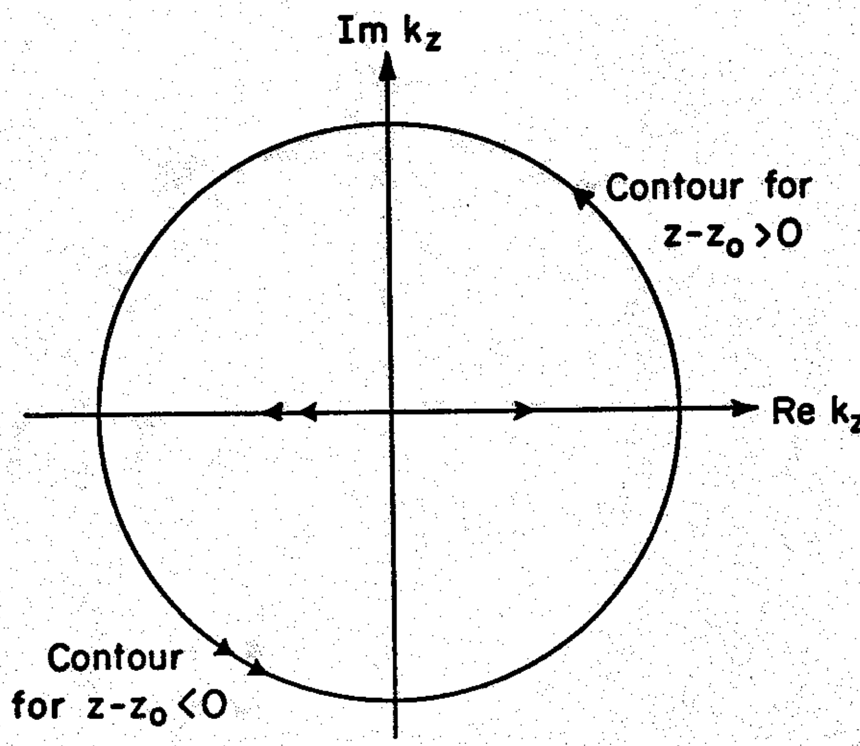


Figure 6 - The k_z Plane

$$\delta j_{\theta}(k,t\to\infty) = \sin(\alpha-\theta)\frac{k_z k_{\perp}\omega_p{}^2 v_o \rho_b(\underline{k})}{D_o} \tag{A.16b}$$

$$\delta j_z(\underline{k},t\to\infty) = -\frac{\omega_p{}^2 k_{\perp}{}^2 v_o \rho_b(\underline{k})}{D_o} \tag{A.16c}$$

$$\delta E_r(\underline{k},t\to\infty) = -\frac{ik\ v_o{}^2\rho_b(\underline{k})\cos(\alpha-\theta)\left[k_z{}^2\left(1+\frac{i}{k_z v_o \tau}\right)+\frac{\omega_p{}^2}{c}\right]}{\varepsilon_o D_o} \tag{A.17a}$$

$$\delta E_{\theta}(\underline{k},t\to\infty) = -\frac{ik_{\perp} v_o{}^2\rho_b(\underline{k})\sin(\alpha-\theta)\left[k_z{}^2\left(1+\frac{i}{k_z v_o \tau}\right)+\frac{\omega_p{}^2}{c^2}\right]}{\varepsilon_o D_o} \tag{A.17b}$$

$$\delta E_z(\underline{k},t\to\infty) = -\frac{ik_z v_o{}^2\rho_b(\underline{k})\left[k_z{}^2\left(1+\frac{i}{k_z v_o \tau}\right)+\frac{\omega_p{}^2}{c^2}\right]}{\varepsilon_o D_o} \tag{A.17c}$$

$$\delta B_r(\underline{k},t\to\infty) = -\sin(\alpha-\theta)\ \frac{ik\ v_o \omega_p{}^2\rho_b(\underline{k})}{\varepsilon_o c^2 D_o} \tag{A.18a}$$

$$\delta B\ (\underline{k},t\to\infty) = \cos(\alpha-\theta)\frac{ik\ v_o \omega_p{}^2\rho_b(\underline{k})}{\varepsilon_o c^2 D_o} \tag{A.18b}$$

$$\delta B_z(\underline{k},t\to\infty) = 0 \tag{A.18c}$$

$$\delta\rho(\underline{k},t\to\infty) = \rho_b(\underline{k})\frac{\omega_p{}^2}{D_o}\left\{\left(k^2-\frac{v_o{}^2}{c^2}k_z{}^2\right)+\frac{p}{\gamma_o{}^2c^2\left(1+\frac{i}{k_z v_o \tau}\right)}\right\} . \tag{A.19}$$

D_o, given in the text in Eq. (3.28), can be factored exactly, the result being

$$D_o = \frac{v_o{}^2}{1+\frac{i}{k_z v_o \tau}}\ (k_z-k_1)(k_z-k_2)\left(k^2\left[1+\frac{i}{k_z v_o \tau}\right]+\frac{\omega_p{}^2}{c^2}\right) , \tag{A.20}$$

where

$$k_{1,2} = -\frac{i}{2v_o\tau} \pm \left(\frac{\omega_p^2}{\gamma_o^2 v_o^2} - \frac{1}{4v_o^2\tau^2}\right)^{\frac{1}{2}} . \qquad \text{(A.21)}$$

The uniform beam charge density (Eq. (3.30)), for which we wish to obtain the spatial variation of the quantities in Eqs. (A.16)-(A.19), has the Fourier transform

$$\rho_b(\underline{k}) = -2\pi n_b eb \frac{J_1(k_\perp b)}{k_\perp} \int_{-\infty}^{o} dz_o e^{-ik_z z_o} ,$$

J_m is the Bessel function of the first kind and order m. In obtaining this result, we have made use of Eqs. (A.15) and the following Bessel function integral representations:

$$\frac{1}{2\pi}\int_o^{2\pi} e^{ik_\perp r \cos(\alpha-\theta)} d\alpha = J_o(k_\perp r) \qquad \text{(A.23a)}$$

$$\frac{1}{2\pi}\int_o^{2\pi} e^{ik_\perp r \cos(\alpha-\theta)} \cos(\alpha-\theta) d\alpha = \frac{1}{ik_\perp}\frac{\partial}{\partial r} J_o(k_\perp r) = iJ_1(k_\perp r). \qquad \text{(A.23b)}$$

The inverse Fourier transform for the quantity δQ, from Eq. (3.29) is

$$\delta Q(\underline{x}, t\to\infty) = \int_o^\infty \frac{k_\perp dk_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dk_z \int_o^{2\pi} d\alpha e^{ik_\perp r \cos(\alpha-\theta) + ik_z z} \delta Q(\underline{k}, t\to\infty) . \qquad \text{(A.24)}$$

With the help of

$$\frac{1}{2\pi}\int_o^{2\pi} d\alpha \sin(\alpha-\theta) e^{ik_\perp r\cos(\alpha-\theta)} = \frac{1}{ik_\perp r}\frac{\partial}{\partial\theta} J_o(k_\perp r) = 0, \qquad \text{(A.25)}$$

we immediately obtain

$$0 = \delta j_\theta(\underline{x},t\to\infty) = \delta E_\theta(\underline{x},t\to\infty) = \delta B_r(x,t\to\infty). \qquad (A.26)$$

Eqs. (A.23) imply that the remaining quantities are

$$\delta j_r = -n_b e b \omega_p^2 v_o \int_0^\infty \frac{k_\perp dk_\perp J_1(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{o} dz_o \int_{-\infty}^{\infty} dk_z \frac{ik_z e^{ik_z(z-z_o)}}{D_o} \qquad (A.27a)$$

$$\delta j_z = n_b e b \omega_p^2 v_o \int_0^\infty \frac{k_\perp^2 J_o(k_\perp r)J_1(k_\perp b)dk_\perp}{2\pi} \int_{-\infty}^{o} dz_o \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}}{D_o} \qquad (A.27b)$$

$$\delta E_r = -\frac{n_b e b v_o^2}{\varepsilon_o} \int_0^\infty \frac{k_\perp dk_\perp J_1(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{o} dz_o \times \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}\left[k_z^2\left(1+\frac{i}{k_z v_o \tau}\right)+\frac{\omega_p^2}{c^2}\right]}{D_o} \qquad (A.27c)$$

$$\delta E_z = \frac{n_b e b v_o^2}{\varepsilon_o} \int_0^\infty \frac{dk_\perp J_o(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{o} dz_o \times \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)} ik_z\left[k_z^2\left(1+\frac{i}{k_z v_o \tau}\right)+\frac{\omega_p^2}{c^2}\right]}{D_o} \qquad (A.27d)$$

$$\delta B_\theta = \frac{n_b e b v_o \omega_p^2}{\varepsilon_o c^2} \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} J_1(k_\perp r)J_1(k_\perp b) \int_{-\infty}^{\infty} dz_o \int_{-\infty}^{\infty} \frac{e^{ik_z(z-z_o)}}{D_o} dk_z \qquad (A.27e)$$

$$\delta\rho = -\frac{n_b e b \omega_p^2}{\gamma_o^2} \int_0^\infty \frac{dk_\perp J_o(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{o} dz_o \int_{-\infty}^{\infty} dk_z\, e^{ik_z(z-z_o)} \frac{1}{D_o}\left[k_z^2 + \frac{\omega_p^2}{c^2\left(1+\frac{i}{k_z v_o \tau}\right)} + \gamma_o^2 k_\perp^2\right]. \qquad (A.27f)$$

The arguments of all of these functions are $(\underline{x},t\to\infty)$. Since the k_z integrals required in δj_r and δE_z are the z derivatives of those required in δj_z and δE_r, respectively, we have only three different k_z integrations to do. They are easily done by contour integration and the Residue Theorem. Restricting ourselves to a weakly collisional plasma, we stipulate

$$\omega_p \tau >> 1 \tag{A.28}$$

and drop terms of order $\frac{1}{\omega_p \tau}$ compared to 1. However, in k_1 and k_2 we must keep the imaginary parts in order to properly locate these poles and to damp the resultant residues. In this case, D_o has roots at k_3 and k_4 given by

$$k_{3,4} = \pm i \left(k_\perp^2 + \frac{\omega_p^2}{c^2} \right)^{\frac{1}{2}} \tag{A.29}$$

to go with k_1 and k_2. Since we are concerned only with current neutralization, we have dropped a pole due to finite "collision" time which, as pointed out to us by R. E. Lee and R. N. Sudan,[22] results in the slow decay of the current neutralization. However, this effect occurs on a length scale greater than beam lengths in the high current beam experiments to date. (See the discussion at the end of Section III.) With reference to Figure 6, for $z-z_o > 0$, we must complete the contour in the upper half k_z plane in order to have convergence on the "infinite circle." This contour, marked with single arrows in Figure 6, includes only the pole at $k_z = k_3$. For $z-z_o < 0$, the contour must be completed in the lower half k_z plane, as indicated by the double arrows in Fig. 6. This contour encloses the three poles, $k_z = k_1, k_2, k_4$. Dropping terms of order $1/\omega_p \tau$, the k_z integrations are

$$\int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}}{D_o} \simeq -\frac{\pi \exp\left[-\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]}{\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}, \quad z-z_o > 0$$

$$\simeq \frac{2\pi}{v_o^2}\left(\frac{\gamma_o v_o}{\omega_p}\,\frac{e^{\frac{z-z_o}{2v_o\tau}}\sin\frac{\omega_p}{\gamma_o v_o}(z-z_o)}{k_\perp^2+\frac{\omega_p^2}{v_o^2}} - \frac{\exp\left[\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]}{2\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}\right), \quad z-z_o < 0 \qquad \text{(A.30a)}$$

$$\int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}\left(k_z^2+\frac{ik_z}{v_o\tau}+\frac{\omega_p^2}{c^2}\right)}{D_o} \simeq \frac{\pi k_\perp^2 \exp\left[-\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]}{v_o^2\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}, \quad z-z_o > 0$$

$$\simeq \frac{2\pi k_\perp^2}{v_o^2}\left(\frac{\gamma_o \omega_p}{v_o}\,\frac{e^{\frac{z-z_o}{2v_o\tau}}\sin\frac{\omega_p}{\gamma_o v_o}(z-z_o)}{\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)k_\perp^2} + \frac{\exp\left[\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]}{2\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}\right), \quad z-z_o < 0 \qquad \text{(A.30b)}$$

$$\int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}\left(k_z^2+\frac{\omega_p^2}{c^2}+\gamma_o^2 k_\perp^2\right)}{D_o} \simeq \frac{\pi}{v_o^2}\,\frac{\exp\left[-\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]k_\perp^2(1-\gamma_o^2)}{\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}, \quad z-z_o > 0$$

$$\cong \frac{2\pi}{v_o}\left(\frac{\gamma_o v_o}{\omega_p}\,\frac{\left(\frac{\omega_p^2}{v_o^2}+\gamma_o^2 k_\perp^2\right)}{\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}\,e^{\frac{z-z_o}{2v_o\tau}}\sin\frac{\omega_p}{\gamma_o v_o}(z-z_o)\right. \qquad \text{(A.30c)}$$

$$\left.+\frac{k_\perp^2}{2}(1-\gamma_o^2)\,\frac{\exp\left[\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}(z-z_o)\right]}{\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}}\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}\right), \quad z-z_o < 0$$

The z_o integrals can now be done using standard forms (for example, from Dwight[27]), taking care to break up the interval of integration as demanded by Eqs. (A.30). Dropping terms of order $1/\omega_p\tau$, we obtain

$$\delta j_r = n_b e v_o \frac{b\omega_p^2}{v_o^2}\begin{cases} \frac{1}{2}\frac{\partial}{\partial z}\Gamma_1^1(-z), & z > 0 \\ \frac{\gamma_o v_o}{\omega_p}\sin\frac{\omega_p z}{\gamma_o v_o}\; e^{\frac{z}{2v_o\tau}}\left[\Psi_1^1-\frac{1}{2}\frac{\partial}{\partial z}\Gamma_1^1(z)\right], & z < 0 \end{cases} \qquad \text{(A.31a)}$$

$$\delta j_z = -n_b e v_o \frac{b\omega_p^2}{v_o^2} \begin{cases} \frac{1}{2}\Gamma_0^2(-z), & z > 0 \\ \frac{\gamma_o^2 v_o^2}{\omega_p^2}\left(1 - \cos\frac{\omega_p z}{\gamma_o v_o}\, e^{\frac{z}{2v_o\tau}}\right)\Psi_o^2 + \Sigma_o^2 - \frac{1}{2}\Gamma_o^2(z), & z < 0, \end{cases} \tag{A.31b}$$

$$\delta B_\theta = -\frac{n_b e v_o b \omega_p^2}{\varepsilon_o c^2 v_o^2} \begin{cases} \frac{1}{2}\Gamma_1^1(-z), & z > 0 \\ \frac{\gamma_o^2 v_o^2}{\omega_p^2}\left(1 - \cos\frac{\omega_p z}{\gamma_o v_o}\, e^{\frac{z}{2v_o\tau}}\right)\Psi_1^1 + \Sigma_1^1 - \frac{1}{2}\Gamma_1^1(z), & z < 0, \end{cases} \tag{A.31c}$$

$$\delta E_r = \frac{n_b e b}{\varepsilon_o} \begin{cases} \frac{1}{2}\Gamma_1^3(-z), & z > 0 \\ -\gamma_o^2\left(1 - \cos\frac{\omega_p z}{\gamma_o v_o}\, e^{\frac{z}{2v_o\tau}}\right)\Psi_1^1 + \Sigma_1^3 - \frac{1}{2}\Gamma_1^3(z), & z < 0, \end{cases} \tag{A.31d}$$

$$\delta E_z = \frac{n_b e b}{\varepsilon_o} \begin{cases} -\frac{1}{2}\frac{\partial}{\partial z}\Gamma_0^2(-z), & z > 0 \\ \frac{\gamma_o \omega_p}{v_o}\sin\frac{\omega_p z}{\gamma_o v_o}\, e^{\frac{z}{2v_o\tau}}\,\Psi_0^0 + \frac{1}{2}\frac{\partial}{\partial z}\Gamma_0^2(z), & z < 0, \end{cases} \tag{A.31e}$$

$$\delta\rho = n_b e b \begin{cases} \dfrac{\omega_p^2}{2c^2}\Gamma_0^2(-z), & z>0 \\ \left(1-\cos\dfrac{\omega_p z}{\gamma_o v_o}\, e^{\frac{z}{2v_o\tau}}\right)\left(\dfrac{\omega_p^2}{v_o^2}\Psi_0^0+\gamma_0^2\Psi_0^2\right)+\dfrac{\omega_p^2}{c^2}\left(\Sigma_0^2-\dfrac{1}{2}\Gamma_0^2(z)\right), & z<0. \end{cases} \quad \text{(A.31f)}$$

Γ_n^{ℓ}, Σ_n^{ℓ} and Ψ_n^{ℓ} are the $k_\perp$ integrals defined by

$$\Gamma_n^{\ell}(\pm z) \equiv \Gamma_n^{\ell}(r;b;\pm z) \equiv \int_0^\infty \frac{k_\perp^{\ell} dk\ J_n(k_\perp r) J_1(k_\perp b)}{\left(k_\perp^2+\frac{\omega p^2}{c}\right)\left(k_\perp^2+\frac{\omega p^2}{v_o}\right)}\, e^{\mp\left(k_\perp^2+\frac{\omega_p^2}{c^2}\right)^{\frac{1}{2}} z} \quad \text{(A.32a)}$$

$$\Sigma_n^{\ell} \equiv \Sigma_n^{\ell}(r;b) \equiv \int_0^\infty \frac{k_\perp^{\ell} dk_\perp J_n(k_\perp r) J_1(k_\perp b)}{\left(k_\perp^2+\frac{\omega p^2}{v_o^2}\right)\left(k_\perp^2+\frac{\omega p^2}{c^2}\right)}, \quad \text{(A.32b)}$$

$$\Psi_n^{\ell} \equiv \Psi_n^{\ell}(r;b) \equiv \int_0^\infty \frac{k_\perp^{\ell} dk_\perp J_n(k_\perp r) J_1(k_\perp b)}{\left(k_\perp^2+\frac{\omega_p^2}{v_o^2}\right)}. \quad \text{(A.32c)}$$

The integrals Σ_n^{ℓ} and Ψ_n^{ℓ} can be done exactly by a contour integration method described by G. N. Watson in A Treatise on the Theory of Bessel Functions,[25] on pages 428-429. For the Γ_n^{ℓ}, we will have to settle for a bound of order $e^{-\frac{\omega_p}{c}|z|}$. We note that since

$\Gamma_n^{\ell}(r;b;0) = \Sigma_n^{\ell}(r;b)$ and $\frac{\partial}{\partial z}\Gamma(-z)\Big|_{z=0} = \frac{\partial}{\partial z}\Gamma(z)\Big|_{z=0}$, all of

the plasma response quantities are continuous through $z = 0$. The relevant Ψ_n^{ℓ} and Σ_n^{ℓ} are

$$\Psi_0{}^0 = \begin{cases} \dfrac{v_o{}^2}{\omega_p{}^2 b}\left[1 - \dfrac{\omega_p b}{v_o} I_o\left(\dfrac{\omega_p r}{v_o}\right) K_1\left(\dfrac{\omega_p b}{v_o}\right)\right], & r < b \\ \dfrac{v_o}{\omega_p} I_1\left(\dfrac{\omega_p b}{v_o}\right) K_o\left(\dfrac{\omega_p r}{v_o}\right), & r > b \end{cases} \qquad \text{(A.33a)}$$

$$\Psi_1{}^1 = \begin{cases} I_1\left(\dfrac{\omega_p r}{v_o}\right) K_1\left(\dfrac{\omega_p b}{v_o}\right), & r < b \\ I_1\left(\dfrac{\omega_p b}{v_o}\right) K_1\left(\dfrac{\omega_p r}{v_o}\right), & r > b \end{cases} \qquad \text{(A.33b)}$$

$$\Psi_0{}^2 = \begin{cases} \dfrac{\omega_p}{v_o} I_o\left(\dfrac{\omega_p r}{v_o}\right) K_1\left(\dfrac{\omega_p b}{v_o}\right), & r < b \\ -\dfrac{\omega_p}{v_o} I_1\left(\dfrac{\omega_p b}{v_o}\right) K_o\left(\dfrac{\omega_p r}{v_o}\right), & r > b \end{cases} \qquad \text{(A.33c)}$$

$$\Sigma_0{}^2 = \frac{\gamma_o{}^2 v_o{}^2}{\omega_p{}^2} \begin{cases} \dfrac{\omega_p}{c} I_o\left(\dfrac{\omega_p r}{c}\right) K_1\left(\dfrac{\omega_p b}{c}\right) - \dfrac{\omega_p}{v_o} I_o\left(\dfrac{\omega_p r}{v_o}\right) K_1\left(\dfrac{\omega_p b}{v_o}\right), & r < b \\ -\dfrac{\omega_p}{c} I_1\left(\dfrac{\omega_p b}{c}\right) K_o\left(\dfrac{\omega_p r}{c}\right) + \dfrac{\omega_p}{v_o} I_1\left(\dfrac{\omega_p b}{v_o}\right) K_o\left(\dfrac{\omega_p r}{v_o}\right), & r > b \end{cases} \qquad \text{(A.33d)}$$

$$\Sigma_1{}^1 = \frac{\gamma_o{}^2 v_o{}^2}{\omega_p{}^2} \begin{cases} I_1\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) - I_1\left(\frac{\omega_p r}{v_o}\right) K_1\left(\frac{\omega_p b}{v_o}\right) , & r < b \\ I_1\left(\frac{\omega_p b}{c}\right) K_1\left(\frac{\omega_p r}{c}\right) - I_1\left(\frac{\omega_p b}{v_o}\right) K_1\left(\frac{\omega_p r}{v_o}\right) , & r > b \end{cases} \tag{A.33e}$$

$$\Sigma_1{}^3 = \Psi_1{}^1 - \frac{\omega_p{}^2}{c^2} \Sigma_1{}^1 . \tag{A.33f}$$

Now consider $\Gamma_0{}^2$. Since $|J_o(k_\perp r)J_1(k_\perp b)| < 1$, $J_0 J_1 - 1 < 0$ and $J_0 J_1 + 1 > 0$. It is clear that

$$\Gamma_0{}^2(\pm z) < \int_0^\infty \frac{k_\perp{}^2 dk_\perp [J_0(k_\perp r)J_1(k_\perp b) + 1] \, e^{-\left(k_\perp{}^2 + \frac{\omega_p{}^2}{c^2}\right)^{\frac{1}{2}} |z|}}{\left(k_\perp{}^2 + \frac{\omega_p{}^2}{c^2}\right)\left(k_\perp{}^2 + \frac{\omega_p{}^2}{v_o{}^2}\right)}$$

$$\Gamma_0{}^2(\pm z) > \int_0^\infty \frac{k_\perp{}^2 dk_\perp [J_0(k_\perp r)J_1(k_\perp b) - 1] \, e^{-\left(k_\perp{}^2 + \frac{\omega_p{}^2}{c^2}\right)^{\frac{1}{2}} |z|}}{\left(k_\perp{}^2 + \frac{\omega_p{}^2}{c^2}\right)\left(k_\perp{}^2 + \frac{\omega_p{}^2}{v_o{}^2}\right)} . \tag{A.34}$$

Now we may take out the maximum value for the exponential since the upper integrand is always positive and the lower one is everywhere negative. There results

$$[\Sigma_0{}^2 - M] e^{-\frac{\omega_p}{c}|z|} < \Gamma_0{}^2 < e^{-\frac{\omega_p}{c}|z|} [\Sigma_0{}^2 + M], \tag{A.35}$$

where

$$M \equiv \int_0^\infty \frac{k_\perp{}^2 dk_\perp}{\left(k_\perp{}^2 + \frac{\omega_p{}^2}{c^2}\right)\left(k_\perp{}^2 + \frac{\omega_p{}^2}{v_o{}^2}\right)} \tag{A.36}$$

is a convergent integral since the integrand tends to zero as $k_\perp^{-2}$ as $k_\perp \to \infty$. Better estimates could be made of $\Gamma_0{}^2$ than that represented by (A.35). However, they would not enhance the important point which is that $\Gamma_0{}^2 = 0\,(e^{-\frac{\omega_p}{c}|z|})$. $\Gamma_1{}^1$ can be handled exactly the same way to obtain the same order result. Since $\Gamma_1{}^3 = -\frac{\partial}{\partial r}\Gamma_0{}^2$, and the derivative with respect to r does not affect the z dependence of the integral, we obtain

$$\Gamma_n{}^\ell(\pm z) = 0\left(e^{-\frac{\omega_p}{c}|z|}\right), \quad \binom{\ell}{n} = \binom{1}{1}, \binom{2}{0}, \binom{3}{1} \tag{A.37}$$

These mean that for $|z| > \frac{c}{\omega_p}$, the contribution to the response functions of the $\Gamma_n{}^\ell$'s will be negligibly small. Since $\omega_p\tau \gg 1$ implies $2v_o \gg c/\omega_p$, we drop the $\Gamma_n{}^\ell$ contributions as they add little to the physics of the problem. However, we lose continuity of our plasma response functions in the process. The results to this level of approximation are presented in Section III, Equations (3.31).

Finally, we wish to calculate δv_z. From Eqs. (3.14) and (3.16)

$$\delta v_z = \frac{e}{\gamma_o{}^3 m} \frac{\delta E_z}{(s - ik_z v_o + 1/\tau)} \quad . \tag{A.38}$$

This gives the time asymptotic limit

$$\delta v_z = -\frac{e}{\gamma_o{}^3 m} \frac{v_o \rho_b(\underline{k})}{\varepsilon_o D_o} \left[k_z{}^2 + \frac{\omega_p{}^2}{c\,(1 + \frac{i}{k_z v_o \tau})}\right]. \tag{A.39}$$

Comparing δv_z with δE_r and δE_z in Eq. (A.27), and dropping $0(\frac{1}{\omega_p \tau})$, we obtain

$$\delta v_z = \begin{cases} \dfrac{n_b eb}{\varepsilon_o} \dfrac{e}{\gamma_o{}^3 m v_o} \left[\Sigma_0{}^2 - \dfrac{1}{2} \Gamma_0{}^2 - \gamma_0{}^2 \Psi_0{}^0 \left(1 - \cos \dfrac{\omega_p z}{\gamma_o v_o} e^{\frac{z}{2v_o \tau}} \right) \right], & z < 0 \\ \dfrac{n_b e^2 b}{2\varepsilon_o \gamma_o{}^3 m v_o} \Gamma_0{}^2, & z > 0 \end{cases} \qquad \text{(A.40)}$$

APPENDIX B

In this appendix, we return to Eq. (3.1) in the beam-at-rest frame and solve our initial value problem without taking moments. We intend to show that this method of attack, in the cold plasma limit, gives the same results as found in Section III.

The linearized version of Eq. (3.1) is

$$\frac{\partial \delta f}{\partial t} + \underline{v}\,\frac{\partial \delta f}{\partial \underline{x}} - e(\delta\underline{E}+\underline{v}\times\delta\underline{B})\cdot\frac{\partial f_o}{\partial \underline{p}} = -\,\frac{\delta f - \frac{\delta n}{n} f_o}{\tau} \quad , \tag{B.1}$$

where

$$\delta f = F(\underline{p}) - f_o(\underline{p}). \tag{B.2}$$

The plasma is streaming by the beam with velocity $-v_o\hat{e}_z$ (and electron momentum $-p_o\hat{e}_z \equiv -\gamma_o m v_o\hat{e}_z$), and $v_o >> v_e$, the electron thermal velocity, by assumption. Consequently, for $f_o(\underline{p})$ we take the two mass approximation Maxwellian derived by Watson, Bludman and Rosenbluth[28] for a particle stream with streaming velocity large compared to its thermal velocity:

$$f_o(\underline{p}) = \frac{1}{\gamma_o(2\pi)^{\frac{3}{2}}\,(\gamma_o m v_e)^3}\,\exp\left(-\frac{1}{2(\gamma_o m v_e)^2}\,[p_\perp{}^2 + \frac{(p_z+p_o)^2}{\gamma_o{}^2}]\right) \ . \tag{B.3}$$

The Fourier-Laplace transform of Eq. (B.1), using $\delta f(t=0)=0$ and Eq. (3.17), is

$$(s+i\underline{k}\cdot\underline{v}+1/\tau)\delta f = e\left(\delta\underline{E} - \frac{i}{s}\,\underline{v}\times(\underline{k}\times\delta\underline{E})\right)\cdot\frac{\partial f_o}{\partial \underline{p}} + \frac{\delta n f}{n\tau} \quad . \tag{B.4}$$

The quantities δf, δn, and $\delta \underline{E}$ are now functions of s and $\underline{k}$. The quantities δn, $\delta \underline{j}$ and $\underline{\underline{\sigma}}$ are defined in terms of δf and $\delta \underline{E}$ by

$$\delta n = n \int \delta f d\underline{p} \tag{B.5}$$

$$\delta \underline{j} = -ne \int \underline{v}\, \delta f d\underline{p} \equiv \underline{\underline{\sigma}} \cdot \delta \underline{E}. \tag{B.6}$$

It is more convenient to work with velocity space integrals for δn and $\delta \underline{j}$, so with Watson, Bludman and Rosenbluth[28] we use the two mass approximation for individual electrons within the streaming plasma,

$$p_\perp \simeq \gamma_o m v \quad , \qquad p_z \simeq \gamma_o^{\,3} m v_z \quad , \tag{B.7}$$

to obtain a velocity space Maxwellian, $f(\underline{v})$. The appropriately normalized result is

$$f(\underline{v}) = \frac{\gamma_o}{(2\pi v_e^{\,2})^{3/2}} \exp -\frac{1}{2v_e^{\,2}}[v_\perp^{\,2}+\gamma_o^{\,2}(v_z+v_o)\ ] \tag{B.8}$$

The use of this and Eqs. (B.7) in the equations for δn and $\delta \underline{j}$ give the desired velocity space integrals for δn and $\delta \underline{j}$:

$$\delta n = \frac{ne}{\gamma_o m s}\int \frac{\delta \underline{E}(s+i\underline{k}\cdot\underline{v})-i\underline{k}(\underline{v}\cdot\delta\underline{E})}{s+i\underline{k}\cdot\underline{v}+1/\tau}\cdot\left(\frac{\partial f}{\partial \underline{v}} - \frac{v_o^{\,2}}{c^2}\frac{\partial f}{\partial v_z}\hat{e}_z\right)d\underline{v} + \frac{\delta n}{\tau}\int\frac{f\, d\underline{v}}{s+i\underline{k}\cdot\underline{v}+1/\tau} \tag{B.9}$$

$$\delta j = -\frac{ne^2}{\gamma_o m s}\int \underline{v}\left(\frac{\delta \underline{E}(s+i\underline{k}\cdot\underline{v})-i\underline{k}(\underline{v}\cdot\delta\underline{E})}{s+i\underline{k}\cdot\underline{v}+1/\tau}\right)\cdot\left(\frac{\partial f}{\partial \underline{v}} - \frac{v_o^{\,2}}{c^2}\frac{\partial f}{\partial v_z}\hat{e}_z\right)d\underline{v}$$

$$-\frac{\delta n e}{\tau}\int\frac{f(\underline{v})\ \underline{v}\ d\underline{v}}{s+i\underline{k}\cdot\underline{v}+1/\tau} \equiv \underline{\underline{\sigma}}\cdot\delta E. \tag{B.10}$$

With the substitution $\underline{v} = v' - v_o\hat{e}_z$,

$$f(\underline{v}) = f'(\underline{v}') = \frac{\gamma_o}{(2\pi v_e^2)^{3/2}} \exp\left(-\frac{1}{2v_e^2}\left[v_\perp'^2 + \gamma_o^2 v_z'^2\right]\right) \tag{B.11}$$

and

$$\frac{\partial f}{\partial \underline{v}} - \frac{v_o^2}{c^2}\frac{\partial f}{\partial v_z}\hat{e}_z = -\frac{1}{v_e^2}\underline{v}' f'(\underline{v}'). \tag{B.12}$$

Using the coordinate system defined in Eqs. (A.1) and vector relationships obtainable from Figure 5, we find

$$v_\perp'^2 + \gamma_o^2 v_z'^2 = v_1'^2 + v_2'^2 + v_3'^2 + (\gamma_o^2 - 1)\left(\frac{k_z}{k} v_1' + \frac{k_\perp}{k} v_2'\right)^2. \tag{B.13}$$

Eq. (B.9) can now be written

$$\delta n \left(1 - \frac{1}{\tau}\int \frac{f'(\underline{v})\, d\underline{v}'}{s + ikv_1' - ik_z v_o + 1/\tau}\right) \equiv \alpha_c \delta n =$$

$$-\frac{ne}{\gamma_o m s v_e^2} \times \int \left[\frac{\delta\underline{E}(s + ikv_1' - ik_z v_o) - i\underline{k}(\underline{v}' - v_o\hat{e}_z)\cdot\delta\underline{E}}{s + ikv_1' - ik_z v_o + 1/\tau}\right]\cdot \underline{v}' f'(\underline{v}')\, d\underline{v}'. \tag{B.14}$$

Consider α_c first:

$$\alpha_c \equiv 1 - \frac{\gamma_o}{\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2v_e^2}\left[v_1'^2 + v_2'^2 + v_3'^2 + (\gamma_o^2 - 1)\left(\frac{k_z}{k}v_1' + \frac{k_\perp}{k}v_2'\right)^2\right]}}{(2\pi v_e^2)^{3/2}\,(s + ikv_1' - ik_z v_o + 1/\tau)}\, dv_1' dv_2' dv_3' \tag{B.15}$$

The v_3' integral is just a Gaussian integral and can be done immediately to give $(2\pi v_e^2)^{\frac{1}{2}}$. The v_2' integral can be put in the form of a Gaussian integral by completing the square in v_2' in the exponent. This leaves the v_1' integral, which can be put in non-dimensional form by a simple substitution to give

$$\alpha_c = 1 + \frac{i\gamma_o}{(2\pi)^{\frac{1}{2}} v_e \tau (k_z{}^2+\gamma_o{}^2 k_\perp{}^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{dy e^{-y^2}}{y - \dfrac{i\gamma_o(s-ik_z v_o+1/\tau)}{(2)^{\frac{1}{2}} v_e (k_z{}^2+\gamma_o{}^2 k_\perp{}^2)^{\frac{1}{2}}}} \tag{B.16}$$

We now introduce the Plasma Dispersion Function of Fried and Conte,[29] and its derivative, defined by

$$Z_p(\Lambda) \equiv \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{x-\Lambda}, \tag{B.17a}$$

$$Z_p'(\Lambda) = -\frac{2}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{x e^{-x^2} dx}{x-\Lambda} = -2(1+\Lambda Z_p(\Lambda)). \tag{B.17b}$$

Then

$$\alpha_c = 1 + \frac{i\gamma_o}{(2)^{\frac{1}{2}} v_e \tau (k_z{}^2+\gamma_o{}^2 k_\perp{}^2)^{\frac{1}{2}}} \quad Z_p(\Lambda) = \frac{s-ik_z v_o - \dfrac{Z_p'(\Lambda)}{2\tau}}{s-ik_z v_o + 1/\tau} \tag{B.18}$$

where

$$\Lambda = \frac{i\gamma_o(s-ik_z v_o+1/\tau)}{2^{\frac{1}{2}} v_e (k_z{}^2+\gamma_o{}^2 k_\perp{}^2)^{\frac{1}{2}}}. \tag{B.19}$$

Similar manipulation of the right hand side of Eq. (B.14) gives

$$\delta n = \frac{neZ_p'(\Lambda)}{\alpha_c 2\gamma_o m v_e^2 iks}\left[s\hat{e}_1+\left(ik_\perp v_o - \frac{(\gamma_o^2-1)k_z k_\perp(s-ik_z v_o)}{k_z^2+\gamma_o^2 k_\perp^2}\right)\hat{e}_2\right]\cdot\delta\underline{E} \quad . \qquad (B.20)$$

To obtain $\delta\underline{j} \equiv \underline{\underline{\sigma}}\cdot\delta\underline{E}$ in terms of $Z_p(\Lambda)$ and $Z_p'(\Lambda)$, we need the additional relationships

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} \frac{x^2 e^{-x^2}dx}{x-\Lambda} = -\Lambda\frac{Z_p'(\Lambda)}{2} \, , \qquad (B.21a)$$

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} \frac{x^3 e^{-x^2}dx}{x-\Lambda} = \frac{1}{2} -\Lambda^2\frac{Z_p'(\Lambda)}{2} \, . \qquad (B.21b)$$

Changing dummy variable of integration in (B.10) to $\underline{v}'$, doing v_3', v_2' - integrations as described above for α_c, and using Eqs. (B.17) and (B.21), together with a considerable amount of algebra, finally yield

$$\sigma_{11} = -\frac{s\varepsilon_o\omega_p^2 Z_p'(\Lambda)}{2k^2 v_e^2\alpha_c} \qquad (B.22a)$$

$$\sigma_{12} = \sigma_{21} = -\frac{\varepsilon_o\omega_p^2 Z_p'(\Lambda) ik_\perp v_o}{2k^2 v_e^2 \alpha_c}\left(\frac{\gamma_o^2 k^2 + \frac{\gamma_o^2}{c^2} ik_z v_o s}{k_z^2+\gamma_o^2 k_\perp^2}\right) \qquad (B.22b)$$

$$\sigma_{22} = \frac{\varepsilon_o\omega_p^2}{s}\left[\frac{k^2\left(\alpha_c+\frac{Z_p'(\Lambda)}{2}\right)}{k_z^2+\gamma_o^2k_\perp^2} + \frac{Z_p'(\Lambda)}{2k^2v_e^2\alpha_c}\left(k_\perp^2 v_o^2 - \frac{(\gamma_o^2-1)^2 k_z^2 k_\perp^2 (s-ik_z v_o)^2}{(k_z^2+\gamma_o^2 k_\perp^2)^2} + \frac{2ik_z v_o(\gamma_o^2-1)(s-ik_z v_o)k_\perp^2}{k_z^2+\gamma_o^2k_\perp^2}\right)\right] \qquad (B.22c)$$

$$\sigma_{33} = \frac{\varepsilon_o\omega_p^2(s-ik_z v_o)}{s(s-ik_z v_o+1/\tau)}\left(1+\frac{Z_p'(\Lambda)}{2}\right) \qquad (B.22d)$$

$$\sigma_{13} = \sigma_{31} = \sigma_{32} = \sigma_{23} = 0 \qquad (B.22e)$$

In Appendix A, Eqs. (A.13) - (A.14), we obtained the components of $\delta\underline{E}$ and $\delta\underline{j}$ in terms of the components of $\underline{\underline{Y}}$ and, therefore, $\underline{\underline{\sigma}}$. We could substitute Eqs. (B.22) into those results and write out the plasma response functions in complete detail. However, as we are primarily interested in showing that the present method gives the results of Section II, we do not require the entire solution. Therefore, we will write out only $\delta\underline{j}$ in detail, and obtain only $\delta j_z(\underline{x}, t\to\infty)$. The components of $\delta\underline{j}$ are

$$\delta j_1 = \frac{i\rho_b(\underline{k})\omega_p{}^2 Z_p'(\Lambda)}{\alpha_c D_{sv} 2k^3 v_e{}^2}\left[k^2+\frac{s^2}{c^2}+\frac{\omega_p{}^2k^2\left(\alpha_c+\frac{Z_p'(\Lambda)}{2}\right)}{c^2(k_z{}^2+\gamma_o{}^2k_\perp{}^2)}\right] , \tag{B.23a}$$

$$\delta j_2 = -\frac{\rho_b(\underline{k})(k^2+\frac{s^2}{c^2})\omega_p{}^2 k_\perp v_o Z_p'(\Lambda)}{s\alpha_c D_{sv} 2k^3 v_e{}^2}\left(\frac{\gamma_o{}^2k^2+\frac{\gamma_o{}^2}{c^2}ik_z v_o s}{k_z{}^2+\gamma_o{}^2k_\perp{}^2}\right) , \tag{B.23b}$$

$$\delta j_3 = 0 , \tag{B.23c}$$

where

$$D_{sv} = \left(1-\frac{\omega_p{}^2 Z_p'(\Lambda)}{2k^2 v_e{}^2\alpha_c}\right)\left(k^2+\frac{s^2}{c^2}+\frac{\omega_p{}^2k^2\left(\alpha_c+\frac{Z_p'(\Lambda)}{2}\right)}{c^2(k_z{}^2+\gamma_o{}^2k_\perp{}^2)}\right)$$

$$+\frac{\omega_p{}^2 Z_p'(\Lambda)k_\perp{}^2[\gamma_o{}^4v_o{}^2k^4+2ik_z v_o s(\gamma_o{}^2-1)\gamma_o{}^2k^2-s^2(\gamma_o{}^2-1)^2k_z{}^2]}{2k^2c^2v_e{}^2\alpha_c\,(k_z{}^2+\gamma_o{}^2k_\perp{}^2)^2} , \tag{B.24}$$

To obtain the time asymptotic behavior of the current, we use the Final-Value Theorem again and obtain

$$\delta j_1(\underline{k}, t\to\infty) = 0 \tag{B.25a}$$

$$\delta j_2 = -\frac{\rho_b(\underline{k})\omega_p{}^2\gamma_o{}^2 kk_\perp v_o Z_p'(\xi)}{\alpha_c D_{ov} 2v e^2(k_z{}^2+\gamma_o{}^2 k_\perp{}^2)} , \tag{B.25b}$$

where

$$D_{ov} = k^2\left[\left(1 - \frac{\omega_p{}^2 Z_p'(\xi)}{2k^2 v_e{}^2\alpha_c}\right)\left(1 + \frac{\omega_p{}^2\left(\alpha_c + \frac{Z_p'(\xi)}{2}\right)}{c^2(k_z{}^2+\gamma_o{}^2 k_\perp{}^2)}\right)\right.$$
$$\left. + \frac{Z_p'(\xi)\gamma_o{}^4 k_\perp{}^2 v_o{}^2\omega_p{}^2}{2v_e{}^2 c^2\alpha_c(k_z{}^2+\gamma_o{}^2 k_\perp{}^2)^2}\right] , \tag{B.26a}$$

$$\xi = \frac{\gamma_o(k_z v_o + i/\tau)}{2^{\frac{1}{2}} v_e\left(k_z{}^2+\gamma_o{}^2 k_\perp{}^2\right)^{\frac{1}{2}}} \tag{B.26b}$$

And α_c is as in Eq. (B.11) with Λ replaced by ξ and $s = 0$. Using Eq. (A.2) to obtain $\delta j_z(\underline{k}, t\to\infty)$, $\delta j_z(\underline{x}, t\to\infty)$ is given by

$$\delta j_z = \int_0^\infty \frac{k_\perp{}^3 dk_\perp}{(2\pi)^3}\int_{-\infty}^{\infty} dk_z \int_0^{2\pi} \frac{d\alpha e^{i\underline{k}\cdot\underline{x}}[-\rho_b(\underline{k})\omega_p{}^2\ \gamma_o{}^2 v_o Z_p'(\xi)]}{\alpha_c D_{ov} 2v_e{}^2(k_z{}^2+\gamma_o{}^2 k_\perp{}^2)} . \tag{B.27}$$

Taking the uniform beam $\rho_b(\underline{k})$ from Appendix A (Eq. (A.22)), the α integration can be done immediately using (A.23) to give

$$\delta j_z = \frac{\omega_p{}^2 n_b e v_o b}{2\pi}\int_0^\infty k_\perp{}^2 dk_\perp J_1(k_\perp b) J_o(k_\perp r)\int_{-\infty}^{0} dz_o \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z_o)}}{D} , \tag{B.28}$$

where

$$D \equiv \frac{\alpha_c D_{ov}(k_z^2+\gamma_o^2k_\perp^2)2v_e^2}{\gamma_o^2 Z_p'(\xi)} = \frac{2v_e^2\ k^2}{\gamma_o^2 Z_p'(\xi)} \left\{\left[\alpha_c - \frac{\omega_p^2 Z'_p(\xi)}{2k^2v_e^2}\right] \times \left[k_z^2+\gamma_o^2k_\perp^2+\frac{\omega_p^2}{c^2}\left(\alpha_c+\frac{Z'_p(\xi)}{2}\right)\right] + \frac{Z'_p(\xi)\gamma_o^4k_\perp^2v_o^2\omega_p^2}{2v_e^2c^2(k_z^2+\gamma_o^2k_\perp^2)}\right\} . \qquad \text{(B.29)}$$

We suspect that the major contribution to the k_z integration in δj_z will come where D nearly vanishes. Since Z_p and Z_p' are complex valued functions, we require both ReD and ImD near zero. We want δj_z in the weak collision limit -- $\omega_p\tau >> 1$. Therefore we assume $k_zv_o >> \frac{1}{\tau}$ and ξ is nearly real. For reference, we plot the real and imaginary parts of $Z_p(\xi)$ and $Z_p'(\xi)$ for real ξ in Fig. 7. For ξ near zero $\text{Re}Z_p'(\xi) \simeq -2$, and ReD does not vanish:

$$\text{ReD} \simeq -k_\perp^2\left[k_\perp^2v_e^2\left(1+\frac{\omega_p^2}{c^2\gamma_o^2k_\perp^2}\right)+\frac{\omega_p^2}{\gamma_o^2}\right] \qquad \text{(B.30)}$$

For $|\xi| \leq 1$ but away from zero, $\text{Im}Z_p'(\xi)$ is large, so that resonances will be prevented where ReD≃0. For $|\xi| > 1$, $\text{Re}Z_p' > 0$, and the last term in D is positive. The first term cannot be negative if $k^2 > \frac{\omega_p^2}{v_e^2}$ since $|Z_p'| < 2$ and $\alpha_c \sim 1$. Thus, resonances will occur for $k^2 < \frac{\omega_p^2}{v_e^2}$ and $|\xi| \geq 1$, the latter implying the use of the asymptotic expansion for $Z_p'(\xi)$ for $|\xi| > 1$:

$$\frac{Z'_p(\xi)}{2} \sim \frac{1}{2\xi^2} \simeq \frac{v_e^2(k_z^2+\gamma_o^2k_\perp^2)}{\gamma_o^2k_z^2v_o^2}\left(1-\frac{2i}{k_zv_o\tau}\right) . \qquad \text{(B.31)}$$

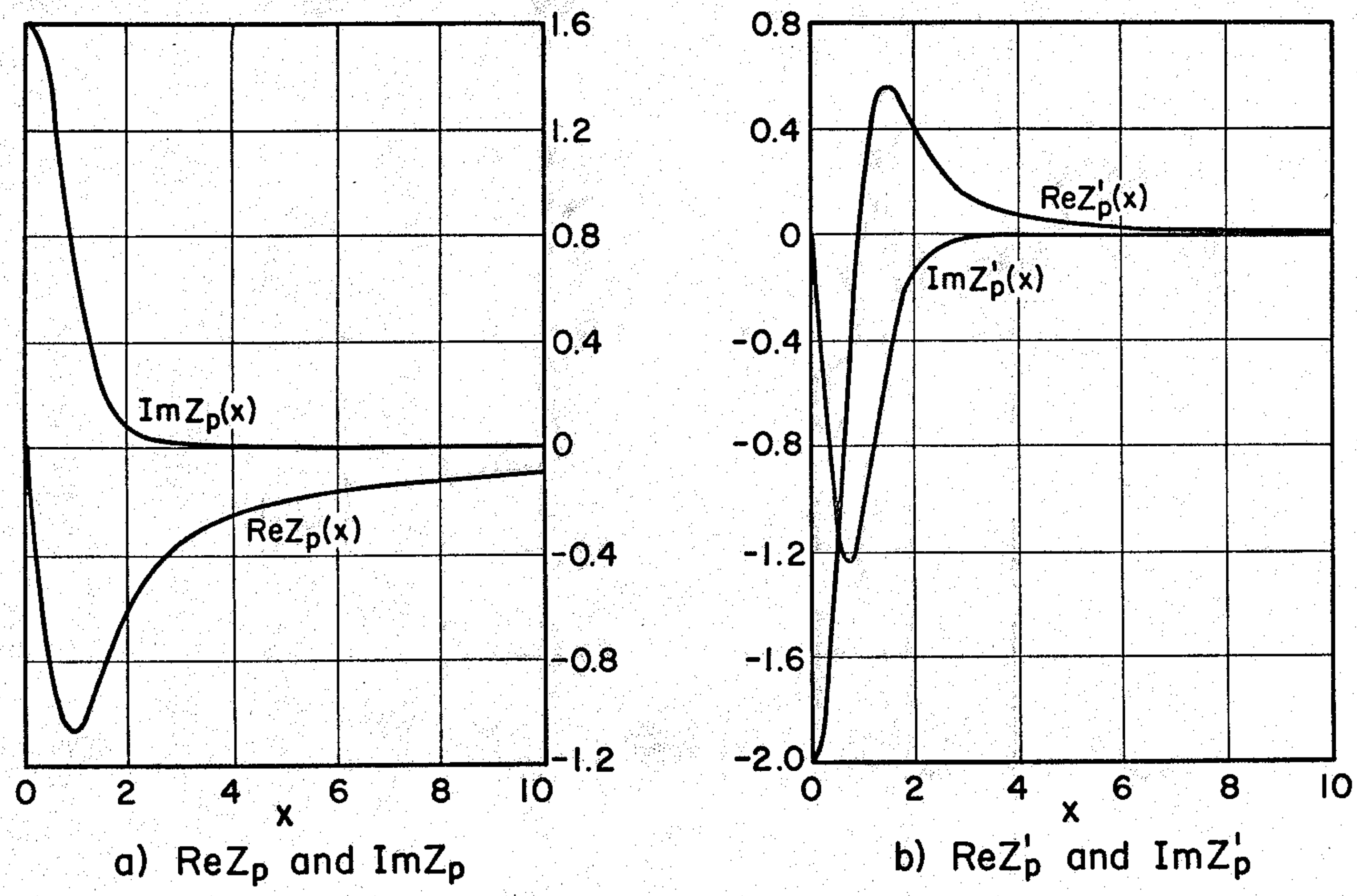


Figure 7

The Plasma Dispersion Function, Z_p, and Its Derivative, for Real Argument[37]

By Eq. (B.18),

$$\alpha_c \simeq \left(1 - \frac{i}{k_z v_o \tau}\frac{Z'p(\xi)}{2}\right)\left(1 - \frac{i}{k_z v_o \tau}\right). \tag{B.32}$$

Expansion of D in this asymptotic limit gives D_{res}, which to highest order in $\frac{i}{k_z v_o \tau}$, is given by

$$\text{Re } D_{res} = v_o^2\left\{k_z^2 k^2 - \frac{\omega_p^2 k^2}{\gamma_o^2 v_o^2}\left(1 + \frac{v_e^2}{c^2}\right) + \frac{\omega_p^2 k_z^2}{c^2} - \frac{\omega_p^4}{\gamma_o^2 v_o^2 c^2}\left(1 + \frac{v_e^2(k_z^2+\gamma_o^2 k_\perp^2)}{\gamma_o^2 v_o^2 k_z^2}\right)\right\}, \tag{B.33a}$$

$$\text{Im } D_{res} = \frac{1}{k_z v_o \tau}\left\{\left(1 - \frac{v_e^2(k_z^2+\gamma_o^2 k_\perp^2)}{\gamma_o^2 k_z^2 v_o^2}\right) k_z v_o^2 k^2 - 2\frac{v_e^2 \omega_p^2 k^2}{c^2 \gamma_o^2} + \frac{\omega_p^4}{\gamma_o^2 c^2}\left(1 + \frac{v_e^2(k_z^2+\gamma_o^2 k_\perp^2)}{\gamma_o^2 k_z^2 v_o^2}\right)\right\}. \tag{B.33b}$$

Since $v_e^2 \ll v_o^2$, c^2 and $|\xi|^2 < 1$, we have

$$D_{res} \simeq v_o^2\left\{k_z^2 k^2\left(1 + \frac{i}{k_z v_o \tau}\right) - \frac{\omega_p^2 k^2}{\gamma_o^2 v_o^2}\left(1 + \frac{2v_e^2}{c^2}\frac{i}{k_z v_o \tau}\right) + \frac{\omega_p^2 k_z^2}{c^2} - \frac{\omega_p^4}{\gamma_o^2 v_o^2 c^2}\left(1 - \frac{i}{k_z v_o \tau}\right)\right\}. \tag{B.34}$$

Except for the remaining term containing v_e^2, to highest order in $\frac{1}{k_z v_o \tau}$, this is the denominator found in the fluid equation calculation.

When we take the limit as $v_e \to 0$, it will be the same denominator. Explicitly, the resonant contribution is

$$\delta j_z^{(res)} = \frac{\omega_p^2 n_b e v_o b}{2\pi} \int_0^{\omega_p/v_e} k_\perp^2 dk_\perp J_1(k_\perp b) J_o(k_\perp r) \int_{-\infty}^{0} dz_o$$

$$\times \int_{|\xi|>1} dk_z \frac{e^{ik_z(z-z_o)}}{D_{res}} \tag{B.35}$$

where

$$|\xi| > 1 \leftrightarrow |k_z| > \frac{2^{\frac{1}{2}} k_\perp v_e}{v_o} \quad . \tag{B.36}$$

In the limit as $v_e \to 0$, $\frac{\omega_p}{v_e} \to \infty$, the k_z integral extends from $-\infty$ to $+\infty$, $D_{res} \to D_o$, and $\delta j_z^{(res)} \to$ fluid equation result, Eq. (A.27b). It is a simple matter to verify that the non-resonant contributions go to zero when v_e goes to zero.

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Laboratory of Plasma Studies Cornell University Ithaca, New York 14850	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Propagation of High-Current Relativistic Electron Beams

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Topical Research Report

5. AUTHOR(S) (First name, middle initial, last name)

David A. Hammer
Norman Rostoker

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June, 1969	90	29

8a. CONTRACT OR GRANT NO. N00014-67-A-0077-0002 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) LPS 16 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Qualified requestors may obtain copies of this report by writing to Director, Laboratory of Plasma Studies, Upson Hall, Cornell University, Ithaca, New York 14850

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Plasma Physics Division Naval Research Laboratory Washington, D.C. 20390

13. ABSTRACT

Theoretical self-consistent beam models have been developed which allow the propagation of relativistic electron fluxes in excess of the Alfvén-Lawson critical-current limit for a fully neutralized beam. Development of a simple, fully relativistic, self-consistent equilibrium is described which can carry arbitrarily large currents at or near complete electrostatic neutralization. A discussion of a model for magnetic neutralization is presented wherein it is shown that large numbers of electrons from a background plasma are counter-streaming slowly within the beam so that the net current density in the system, and therefore the magnetic field, is nearly zero. A solution of an initial-value problem for a beam-plasma system is given which indicates that magnetic neutralization can be expected to occur for plasma densities that are large compared to beam densities. It is found that the application of a strong axial magnetic field to a uniform beam allows propagation regardless of the magnitude of the beam current.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Plasma Physics Relativistic Electron Beam Propagation Magnetically Neutralized Beams Magnetic Guide Fields						